Accademia Gioenia di Catania
Anno di fondazione 1824

Boll. Accad. Gioenia Nat. Sci. (Catania)
56 (386), FP28–FP41 (2023)
doi:10.35352/gioenia.v56i386.108

Many infinities, or only one?*

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Summary

In this short note we first account for Aristotle's views on infinity, by clarifying the way his notion of potential infinity should be understood, in the light of his notion of entelechy. We then present four distinct ways in which mathematicians attempted to tame the notion of actual infinity, and we ask the question of whether they are indeed four different ways, or whether they ultimately are variations on the same concept. We suggest that what mathematicians are doing is indeed to find a way to construct a form of actual infinity that subsumes, into itself, the potential infinity, and observe how the presence of the former brings to a different view about potential infinity than Aristotle's.

Keywords: *Aristotle, Potential and actual infinity, Projective Geometry, Infinitesimal Calculus, Set Theory, Measure Theory.*

*Paper presented by the authors during the ordinary session of January 18, 2023.
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Riassunto

Molti infiniti o uno solo?

In questa breve nota consideriamo dapprima ie idee di Aristotele sull'infinito, chiarendo il modo in cui la sua nozione di infinito potenziale deve essere intesa, alla luce della sua nozione di entelechia. In seguito presentiamo quattro modi distinti in cui i matematici hanno tentato di addomesticare la nozione di infinito attuale e ci chiediamo se si tratti effettivamente di quattro modi diversi o se si tratti in realtà di variazioni dello stesso concetto. Suggeriamo che ciò che i matematici fanno è costruire una forma di infinito attuale che sussuma in sé l'infinito potenziale, e osserviamo come la presenza del primo porti a una visione dell'infinito potenziale diversa da quella di Aristotele.

Parole chiave: *Aristotele, Infinito potenziale e attuale, Geometria proiettiva, Calcolo infinitesimale, Teoria degli insiemi, Teoria della misura.*

1 Introduction

It is almost impossible to find a culture that has not paid attention to infinity. Everything we see, hear, and touch is finite: lots of stars in the sky, lots of molecules of gas in a bottle, lots of grains of sand on the beach, and yet always a finite number, though possibly a very large one. And yet, it is not surprising that from ancient times philosophers have been interested in infinity not only from the point of view of space (is the universe infinitely extended?), or from that of time (is the universe eternal?), but also from other points of view, including that of the first principles. Since this is not strictly a philosophy article, or at least not a note aimed to philosophers only, we will not offer an even cursory summary of how philosophers interpreted this idea, except for a necessary proviso concerning Aristotle's opposition to actual infinity, as something self-subsisting. Indeed, we will try to show that his rejection of actual infinity does not lead, as it is commonly maintained, to potential infinity *sic and simpliciter*.

At first sight, the idea of infinity seems connected with that of quantity. How many stars are there in the sky over us, and how many cells in our body? The answer is certainly a finite number, even if it very unlikely we will ever be able to (precisely) know which finite number. But we could also ask such questions when the answer cannot be given by a finite number, even one that we would not be able to identify or reach with counting. We could ask for example how many positive integer numbers are there? How many real numbers? How many points on a line? How many songs might be sung, or poems might be written, or symphonies might be composed? One cannot answer these questions without speaking of infinity. But what does this mean, exactly? What does it mean to say that there are infinitely many objects of a certain kind? Mathematicians know different ways to answer these questions. In particular, they know how to answer both by plainly recognizing that the number in question is infinite as such, and by observing that no matter how many objects of the relevant kind are considered, other remains to be considered. The former approach involves the so called actual infinity; the latter the so called potential infinity.

But asking how many is not the only way to encounter infinity. This also occurs when asking if something will ever happen (or has happened). An example is given by Zeno's famous paradox of Achilles and the Tortoise. In cases like this, the idea of infinity seems different in nature, since, at least at a first glance, it is not only a question of counting some items (but note that in *Phys*. VIII 8, 263a4-11, while reporting the way "someone" presents the other paradox of the dichotomy, Aristotle interprets it in terms of counting: "in the time during which a motion is in progress we should first count the half-motion for every half-distance that we get, so that we have the result that when the whole distance is traversed we have counted an infinite number"). Still also here mathematics may have a word to say, and it is not at all surprising to discover that mathematicians had in fact been interested in capturing and domesticating this and other ideas of infinity.

When we think of infinity in relation to counting, what we want, presumably, is to be able to associate a number to sets for which no finite (positive integer) number would be appropriate: a number telling us how many distinct elements belong to these sets. But even in the case of the Achilles paradox, we still face some arithmetic steps. In keeping with Lewis Carrol's artistry, we can imagine the Tortoise challenging Achilles and asking him to let her to start closer to the finish line. When Achilles reaches the Tortoise's starting point, his contender has already moved ahead a bit. So Achilles now moves to where the Tortoise has arrived, but at that point, the animal has again moved forward, and so forth endlessly. The mathematician sees, beneath the surface of the problem, an infinite sum and using the notion of limit is able to calculate the total distance that Achilles must cover in order to reach his contender. Just as Achilles can claim to have defeated his opponent, the mathematician can claim to have overcome Zeno's challenge thanks to the appeal to an infinitary operation (the limit) with a finite result.

But, as all human beings, mathematicians have finite bodies and finite minds, and they cannot reproduce infinity as such, and neither can they directly accede to it, and, as Aristotle notices (in *Phys*. III 7, 207b29-31), they do not even need to do so. Aristotle was in fact reasoning only on cases like this, where the appeal to infinite only concerns (or can be reduced to) an endless process: these are cases that mathematicians have learned to describe in term of appropriate functional relations and are often referred to as instances of potential infinity. There are other cases where no such reduction is possible, and the mathematicians are forced to face infinity as such, without appealing to any reduction. These cases are typically described by speaking of actual infinite. Still, also in these cases, they can only somehow mimic infinity with finite instruments, and then hope to control it, or at least improve the way they think about it. As we look at the history of mathematics, we identify, among others, four fundamental moments in which mathematicians have done so, first informally, and later on more and more precisely. We will devote Sections 3–6 to a short illustration of each of these attempts. We will begin, however, with a section in which we attempt to clarify the notion of potential infinity.

This article stems from a discussion of ideas that are being developed in Panza *et al.* (to be published), but offers additional thoughts both with respect to the way in which one can reconcile the work of mathematicians with the ideas expressed by Aristotle on infinity, as well as with respect to some additional comments on measure theory, that are not discussed in Panza *et al.* (to be published).

2 Aristotle and the allegedly potential infinity

Let us come back to the Achilles paradox. To claim that mathematicians have solved it when they have introduced infinite sums—or series, in a more precise mathematical language—and their limits is incorrect, strictly speaking. What series do is merely providing a way to clearly express the solution. Let us clarify this comment, which was already suggested by Aristotle.

In *Physics*, VI 9, 239b14-26, Aristotle describes the paradox by observing that, according to it, "the slowest runner can never [οὐδέποτε] be reached by the quickest, since the pursuer must aforetime [ἔμπροσθεν] reach the point from where the pursued began to move" (we used here and in what follows Hardie and Gaye's translation, with some amendments to make it closer to the Greek text). It is not possible to establish how much Aristotle's testimony is faithful to Zeno's original formulation (Graham, 2010). In particular, it is not possible to know for sure whether Zeno raised the issue of the divisibility of time together with that of the distance. What is certain is that, in reconstructing the argument, Aristotle clearly refers to time, and this in two occasions: when he says that Achilles can "never" (239b15) reach the Tortoise, and when he says that the former must "aforetime" reach the starting point of the latter. Time is also what allows Aristotle to solve the paradox. The solution is given in *Physics*, VI 2, 233a21-28. Strictly speaking, this is a solution to another paradox, that he also attributes to Zeno (239b11-14), according to which a mover cannot reach its destination since it is always possible to bisect the distance. Still, as Aristotle doesn't fail to observe (239b18-20), the two paradoxes are in fact the same, except for the fact that the Achilles paradox involves two movers rather than only one, and that it does not necessarily requires that the distance be reiteratively divided in two equal parts. The solution of the latter also applies then to the former. Here is what Aristotle writes:

> [. . .] Zeno's argument makes an error in asserting that it is not admissible to run across the infinities or to touch them severally in a finite time. For there are two ways in which lenght and time and generally any continuum are said <to be> infinite: they are said <to be> so either in respect of divisibility or in respect of their extremities. Hence, while it is not admissible to touch the infinities in respect to quantity

> in a finite time, this is possible for those in respect to division, for the time itself is infinite in this way.

Aristotle seems here to suggest to look at the race of Achilles and the Tortoise in the other way around: not from the point of view of its course, but from that of its final result. He seems to admit that Achilles actually reaches the Tortoise and then observes that both the distance between the starting point of the former and that in which he reaches the latter, and the time that the two contenders take to arrive there, though finite, are infinitely divisible. Thus Zeno's argument involves no impossibility, since the way the distance is (infinitely) divisible is also the way in which the time is so.

The same point is also made in *Physics*, VIII 8, 263a11-15:

> [...] in our first discussions of motion we solved <the difficulty> through the fact that time contain in itself infinite <elements>: hence there is no absurdity in <the fact that> someone goes through infinite <elements> in infinite time; the infinity is the same in length and time.

All that series do in the solution of the paradox is then providing a way for getting back to the finite distance and time, after the infinite division of them. The core of the solution of the paradox is not there. It is rather in Aristotle's remark that both distance and time (and not only the former) are infinitely divisible.

But in this remark there is more than that, since infinite divisibility can also be seen as the key to understand the infinity of counting. One might indeed understand Aristotle's views about (integer positive) numbers as strictly connected with infinite divisibility itself. Indeed, for him, counting seems primarily to be counting how many times we can divide a continuum, and the idea that, for any number, there is another number greater than it comes exactly from the fact that any part of a continuum can, in turn, be divided into parts. Here is what he writes in *Physics*, III 6, 206b3-6:

> [...] <the infinite> by addition is, in a way, the same as that by division. Since in what is limited, <the infinite> by addition comes about in a way inverse <to that of the other>: since <a magnitude> is seen to be divided *ad infinitum*, in the same way as it appears to be added to what is bounded.

Hence, though there is no limit in counting, there is one in extension, that is, there is no infinitely extended magnitude. Since, by adding the parts we get by division, we cannot but come back to the original (finitely extended) magnitude that we began to divide.

This is the argument that is generally referred to when speaking of Aristotle's view about potential infinity. This last notion requires clarification, however.

There is no doubt, indeed, that, in his analysis of infinity in *Physics* III, Aristotle denies that infinity can exist in act (see *Physics*. III 5, 204a20-21, for instance).

This denial is fundamental for his physics, since it does not only allow him to refute the actual infinity of the Pythagoreans and Platonists, but it also and above all provides the grounds for his refutation of the Eleatics and the Atomists.

For the Eleatics, admitting the divisibility of the One (or the Being) would have gone together with admitting its infinite divisibility. And this for two reasons: first because there might have been no reason for the One could be divisible at one point but not at another (provided that the One cannot admit differences in it), and second because whenever we divide the One, we always have a one after the division (for each of the parts is one), and therefore we can always go ahead in the division, i.e. we can infinitely divide. But, admitting that the One may be infinitely divided would have gone, in turn, together with recognising that it would have been eventually infinitely divided since, for the Eleatics, the possibility of division could not have been explained other than by the admission of a future actual division. This would have then resulted not only in having the same thing both infinitely divided and yet still infinitely divisible—which they would have seen as an absurdity—but also in the other, even more fundamental absurdity of making the One disappear, and with it, also the multiplicity, which would have been, indeed, nothing by a sum of units (Giardina, to be published).

The Atomists were incapable of refuting this argument, since, by admitting the existence of infinite atoms in act, they were merely displacing the problem from the One to the atoms themselves, thus ending up maintaining the indivisibility of the one. More than that, they were incapable to explain in a satisfactory way the atoms indivisibility, by merely appealing for that to their extreme smallness, following Leucippus, or to their fullness, following Democritus. The Atomists, therefore, had no logical reason to contradict the Eleatics in their assuming that, once the divisibility of magnitude is admitted, infinite divisibility follows by logic.

In arguing against these positions, Aristotle first held, in agreement with the Eleatics and in disagreement with the Atomists, that admitting the divisibility is the same as admitting the infinite divisibility. Then he denies the possibility of infinite in act to reject the Eleatics's *reductio*, and remarks that "it then remains that the infinite exists in power [δυνάμει]" (*Phys.* III 6, 206a18).

This passage has been often taken as an evidence for Aristotle's stance in favor of potential infinity. Things are not so straightforward, however. First of all, in saying that, Aristotle is not properly expressing his own view, but only indicating an apparent natural alternative to actual infinity. Moreover, for him, nothing exist that is only in power, and he goes ahead indeed by suggesting a third solution, according to which the reality of the infinite is neither that of being in power, nor that of being in act. There is need of this third solution since infinity can neither exist in act, in the sense of an accomplished condition of an entity, nor only in power, because nothing can exist that is only in power. In order to clarify the nature of the Aristotelian infinite, we will first try to explain the way in which Aristotle speaks of power. Then, after having introduced two different notions of act, we will account for the peculiar nature that Aristotle assigns to the infinite.

To this purpose, let us reflect on an essential aspect of his ontology. Power (δύναμις) individuates the possibility that an entity has of realizing itself as a certain thing that it is (still) not, e.g., it individuates the possibility of a certain piece of bronze of being a certain statue, which it is (still) not. By identifying what an entity can be but is (still) not, power identifies, in other terms the non-being of a thing. By non-being, Aristotle does not mean, indeed, as Parmenides did, absolute non-being, which does not exist for him, but rather the non-being of a certain specific thing, e.g. the non-being of the Charioteer of Delphi for a piece of bronze. For Aristotle, it is then not possible to speak of non-being without referring to a being. This means that for an entity, being potentially something, that is, its non-being, but being able to be this thing, is determined by the being of this very thing. For instance, we can say that this piece of bronze *is not* the Charioteer of Delphi only in force of the Charioteer of Delphi, which this piece of bronze can be, and perhaps will be in future, if nothing prevents it. In Aristotle's ontology, act is then prior to power ontologically speaking, because it is only on the basis of the being that one can admit the non-being: it is only on the basis of the actual hen that one can admit that the egg is potentially a hen. This ontological priority of the act is repetitively made clear in *Metaphysics* IX.

But in which sense, then, can infinite be in power, provided that power requires act, and infinite in act is not possible? Aristotle of course understands the problem very well. Just after the passage quoted above from *Phys*. III 6, 206a18, he observes that "we should not take 'in power' as if we did say 'this thing is a statue in power' and 'as well this thing will be a statue, also the infinite will be in act' (*Phys*. III 6, 206a18-21). The nature of infinite is indeed quite peculiar. On the one side, we cannot apply the same relation between power and act that we apply to the bronze and the statue, for the infinite in power does not entail the possibility of an infinite in act. On the other side, also the infinite cannot exist merely in power, since, for Aristotle, the existence in power is ontologically inadmissible without the possibility of an existence in act.

The solution of the problem comes from a clear indication that Aristotle provides some lines above, in *Physics*. III 6, 206a14-15, where Aristotle observes that "the being <of the infinite> is said to exist both δυνάμει and ἐντελεχείᾳ, and the infinite is both by addition and by division". The term 'ἐντελέχεια', which is coined by Aristotle himself, means a mode of being in act that somehow co-exists with being in power. For example, the piece of bronze that is becoming the Charioteer of Delphi is on the one hand this statue in power, because it is not yet the finished statue, but it is also, on the other hand, already this statue, partially, since it is acquiring the form of the Charioteer, and it is no longer the raw bronze stored in the sculptor's workshop before he begins working on it.

When Aristotle excludes infinity in act, he means that it cannot be something self-subsistent: this is indeed the concept of act that Aristotle expresses with the term 'ἐνέργεια', also coined by him. But he also argues that the infinite is both δυνάμει and ἐντελεχείᾳ, by so identifying its nature with a coincidence of power and another sort of act, which is just the ἐντελέχεια. This makes the being of the infinite consist in its being able to be. In other words, the being of infinite is, for Aristotle, its always becoming something else, like the day that ceaselessly succeeds to another day (*Physique* III 6, 206a22).

This identity of power and act (where act does not mean the condition of a self-standing entity, like Zeno's infinite in act) determine the nature of the infinite: it consists in the fact that the divisibility of a magnitude never ceases, and never gets completion. So by infinitely dividing a magnitude, the number by which the parts are counted increases. Like in *Physics* III 6, 206a14-15 (quoted above), also in *Methaphysics* IX 6, 1048b15-17, Aristotle argues that "the fact that division never ceases to be possible makes this actuality exist potentially, but not it exist separately" (Giardina, 2012; Bowin, 2007; Lear, 1979-80).

It is then far from accurate to simply say that, for Aristotle, infinite is only potential. It would rather be more precise to say that act and power coincide in infinite. This excludes both the possibility of an infinite in act, and that of an end of divisibility and counting. In a sense, also Aristotle infinite is, then, in act as those that is now time to deal with.

3 Painting and Projective Geometry

Let us come back to mathematics by beginning with the infinite of the painters. They must have been always marveled at the complexity of translating what they saw, into what they painted. Even without being a painter, we cannot avoid being surprised when looking at two parallel railroad tracks, and realize that they appear to eventually intersect. But where do they intersect? At infinity, could be a somewhat glib answer. But no so glib, in fact, when mathematicians, or mathematically cultivated painters (as Piero della Francesca and Leonardo) decided to understand this issue in depth. The ideas developed by such painters are indeed the foundation for a new geometry: a geometry of painting (and, today, of computer graphics, too) that goes under the name of projective geometry.

This geometry deals with points and lines at infinity. In our everyday vision, these are points and lines that we see at the horizon, when we see the railroad tracks concur, or the sky reach the sea. Of course, these points do not exist as physical objects: not at the end of the tracks, nor at the intersection between sea and sky, an intersection that does not exist. But the painter has to locate them on the canvas, if the picture is to reproduce what we see. And not at an infinite distance from the bystander's eyes, but, very often, in the very center of the scene, in an eye of the Christ, possibly, as in Leonardo's *Last Supper*.

The role of Renaissance painters in understanding a new role for infinity is the subject of a very influential work of Field (1997), where however there is no discussion of what we consider to be a major step in understanding this challenge, namely the ability of mathematicians to find way to give coordinates to these points and lines, very similarly to how traditional Cartesian coordinates can be used to represent the points on a plane or in space. Once this coordinate assignment is done, there is no difference between the usual points, and those points that are supposed to be infinitely far away. Projective geometry, besides offering a very clear and rather simple procedure allowing us to represent everyday life scenes on a canvas, or on a computer screen, has effectively tamed infinity, by reducing those distant points to points with simple numerical coordinates. Like it often happens in mathematics, the first steps in this direction were uncertain, clumsy, and incomplete. But within a couple hundred years a full theory emerged: rigorous, clear, practically applicable (the details of how this developed can be found in Gentili *et al.* (2023), as well as in Panza *et al.* (to be published). And, like it also often happens, this theory turned out to be much more than simply a way to solve a practical problem of painters, and it opened the way to the understanding that geometry can tell us much more not only than what the mere etymology of its name (originally meaning '[art of] measuring the earth') suggests, but also than what is suggested by the study of some fundamental figures and curves. When we look at at projective geometry we begin to see that it rather deals with a plurality of structures, as it will become manifest a few centuries later.

What had happened, therefore, is the discovery that there are in fact many different geometries, whose applications, properties, and results are dramatically different from one another, and that projective geometry can provide the common framework for the three metric geometries of Euclid, Bolyai-Lobachevskij-Gauss, and Riemann.

4 From the Eleatic Paradoxes to Infinitesimal Calculus

Geometry did not wait for these developments to tackle problems that cannot be solved by merely drawing lines and circles, as taught by Euclid in the first books of the *Elements* (whose title displays their elementary nature, but hides their deductive profoundness and logic sophistication). The Greek philosophers already posed several interesting questions, sometimes in the form of paradoxes—as for the mentioned case of Achilles and the tortoise—that stimulated efforts to understand the nature of space and time, and whether the universe is discrete or continuous (Heath, 1961). In this context, for example, Zeno introduced another paradox concerned with the shooting of an arrow (Vlastos, 1966; Shamsi, 1994). An archer shoots an arrow to hit a target, but the philosopher explains why this will never happen: indeed the arrow cannot move in any of the instants in which time is divided, because motion does not happen unless time is flowing. Since there is no speed without time, and every instant is timeless, by consequence, there is no motion.

More or less at the same time, Greek mathematicians were also struggling with the attempt to calculate areas enclosed by curves that could only be approximated by polygons. A sound method to achieve this goal was accomplished only with the invention of differential and integral calculus more than 2.000 years after the problem.

The invention of calculus (Jahnke, 2003) finally allowed humankind to formulate a theory that would, on one hand, explain why the arrow has velocity even when we freeze time and look at a specific instant, and, on the other, allow us to calculate areas of curvilinear figures, as well as volumes of curved solids. But maybe the most impressive discovery was the fact that calculus shows that these two questions are simply the two faces of a single medal: on one face the calculus of areas, on the other face the construction of tangents and instantaneous speeds. It took the genius of Newton to connect to each another these apparently totally unrelated questions (Panza, 2012), by so providing the base for the most powerful and versatile mathematical theory, whose applications go from the most elementary parts of mathematics, as arithmetic itself, to the most entangled aspects of natural, social and biomedical sciences.

By calling this theory with its usual name, 'infinitesimal analysis', we still pay tribute to its origins, where the idea of a punctual rate of change or instantaneous speed connected with that of infinitesimally small interval of time or space, with Newton and Leibniz sharing the glory of such a momentous achievement. Some two hundred more years were required to see how the theory can be freed from infinitesimally small magnitudes, or actual infinitesimals, in favor of infinitely protracted processes or potential infinity. Processes as those already envisaged by Eudoxus, a pupil of Plato, the greatest mathematician of the golden age of Athenian science and culture, in its exhaustion method. The climax of these other two hundred years was a new version and proof of Newton's basic result, where areas and tangents (or speeds) are replaced by appropriate operators, namely derivatives and integrals. The way these operators connect is displayed by the so called Fundamental Theorem of Calculus (for an exposition of the basic elements of the calculus, see, among many others, Rudin, 1964).

5 Set Theory

At the same time as mathematicians begun to explore the consequences of their new way to look at infinitesimal analysis, they encountered questions whose mere statement required new ways to count. In this new framework, continuity ceased to be a property of magnitudes (or, more generally, of space and time), to become one of functions. More than that, the question became of describing where functions were continuous, and to explain what one meant by saying that a function was continuous at 'all points of a (real) interval.' More crucially, the question became to describe how to count the points in which some function are discontinuous.

Cantor was (one of) the first to see that stating these question, and having a chance to answer them, depended on mastering the apparently simple notion of a set. In so doing, Cantor ushered a new era, by opening the door for a new conception of mathematics as an autonomous discipline that does not require any external reference but merely studies abstract objects (sets) independently defined. Doing so was a process that began in the last quarter of 19th-century, and continued at least throughout the first half on the 20th century (in some aspects, the process has still not reached its conclusion). This shows, at a minimum, that the notion of set is more powerful and complex than Cantor and his contemporaries might have thought.

We cannot, in a short note, discuss any specific detail, but we will offer a couple of comments that hopefully will shed some light on the magnitude of the challenges. A simple way to approach this task is to ask whether the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is as big as the set of even numbers $\mathbb{P} = \{0, 2, 4, 6, \ldots\}$. On one hand, the answer seems trivial. Of course $\mathbb{N}$ is bigger than $\mathbb{P}$, because it contains it properly. On the other hand one can establish a correspondence (to each positive integer its double) that seems to show that there are as many positive integers as there are even integers. This apparent oddity becomes even more unpleasant when one realizes that it is possible to claim (with a similar argument) that there are as many positive integers as there are fractions (or rational numbers as mathematicians say). This surely seems impossible, and yet it is. Does this mean that all infinite sets are essentially equally large? The answer was provided by Cantor, who showed (with a beautiful proof) that in fact there are larger infinities, for example that the set $\mathbb{R}$ cannot be put in correspondence with $\mathbb{N}$ and it is therefore larger. The next step, also achieved by Cantor, is to show that there are in fact infinitely many different kinds of infinities, each one larger than the previous one. This is what Hilbert called Cantor's paradise; whether in fact it is a paradise, or a purgatory, is something we will not discuss here.

The next big challenge, however, arises when trying to understand what the word 'set' means. One is certainly justified in thinking that sets can be simply taken as collections of objects, and that they can be defined through the properties that those objects must satisfy (so that we have the set of Italian citizens, or the set of students in a class, or the set of even numbers, etc.). But with a strikingly simple example, Bertrand Russel shuttered this illusion and made clear the necessity of a very different way to deal with sets, namely an axiomatic theory. Under this theory, sets are not explicitly defined, but axioms are given that allow us to make the sets necessary for the reconstruction of most mathematics available. The most important such formalization was due to Zermelo (1930) and goes under the name of ZFC, where the initials stand for Zermelo, Frankel, and the Axiom of Choice (an apparently strange axiom that allows us to make infinite choices, even in the absence of specific criteria, an idea that is not as simple as it may appear (Fraenkel *et al.*, 1945). From our point of view, the importance of these axioms is the fact that they allow the existence of infinite sets (actual objects containing within them the potential infinity of natural numbers), and then providing, as it were, a context in which the infinite reiteration of counting, division, etc. can take place). The existence of such infinite sets transforms the discussion about the nature of infinite. While, as we pointed out in Section 2, Aristotle could not agree to a genuine concept of potential infinite built in his general notion of power, because power, in this sense requires existence in act, and thus developed the concept of entelechy, we see now a good example of infinite in act. Thus, for modern mathematicians, infinite in act exists, and thus it justifies our comment that such infinite contains within it (and provide a context for) a potential infinite in a genuine sense.

6 Measure Theory

Measure theory deals with an apparently very simple question: how do we measure the size, so to speak, of geometrical objects? We instinctively know what is the measure of a segment, its length, of a polygon, its area, of a solid, its volume. But can we make such an instinctive (and naive) approach work in more complex cases? Here we encounter similar challenges to those we met when working on set theory. A concept that appears to be absolutely commonsense becomes very difficult when one tries to be precise, and the problem of measure is something that is still been considered attentively by working mathematicians, as indicated for example at the very beginning of Tao (2021). One could start with simple properties that a measure should satisfy. For example, the measure of the empty set should be zero, the measure of the union of two disjoint figures (whether in the plane or in space) should be the sum of their measures, and measure should be invariant by translations (in other words, it should not matter, when we measure the volume of a solid, for example, where the solid is located). Even under these simple constraints, problems quickly arise. For example set theory tells us that the intervals $[0, 1]$ and $[0, 2]$ can be put in one-to-one correspondence, and yet we would like to say that the measure of the first is 1 and the measure of the second is 2. What is happening is very disturbing: it shows that we can take the original interval $[0, 1]$, rearrange cleverly its infinitely many points, and recreate a new interval, $[0, 2]$, whose measure is twice the original one. Set theory, however, does even worse than this. For example, the apparently innocuous Axiom of Choice allows the so-called Banach-Tarsky paradox (Banach and Tarski, 1924), that states that it is possible to take a radius one solid sphere in dimension three, split it into five pieces, and then rearrange those five pieces (only using translations and rotations, nothing fancy) to reconstruct two identical radius one spheres. In other words, the notion of measure is so odd, that I can take a sphere, split it into a (small) number of pieces, and almost magically duplicate its volume. It is clear that discoveries such as this cast a pall on the entire enterprise, and leave mathematicians with two unpleasant choices: get rid of the Axiom of Choice (something most mathematicians would hate to do for many very concrete reasons) or accept that there are sets that cannot be measured (in particular the five pieces in which we split the sphere in the Banach-Tarsky paradox are such kinds of sets), Mathematicians decided to accept the existence of non-measurable sets, and this led them to what is currently the standard measure theory, known as Lebesgue measure theory.

7 Conclusions

Let us see what we can conclude from these short remarks. Does this story have an end? And what about these four infinities: are them different aspects of a single notion, or distinct notions?

Before addressing these questions, we would like to make a couple of remarks. To begin with, we hope it is clear that none of these four theories is strictly speaking a theory of infinity, and in particular none of these theories attempts to define the notion of infinity as such (and no other mathematical theory tries to do so either). Rather, they describe objects and/or deal with processes that are, in a sense or another, infinite. We note that set theory is a foundational theory, designed to provide the building blocks for all of mathematics, and certainly at least for the other theories described in this essay. As such, this is the only theory (among those described here) that is explicitly presented as an axiomatic system, and the only one where we explicitly speak of infinite objects, where in fact several different definitions can be given for a set to be infinite. Let us now go back to the questions we have posed.

The first answer is simple. Our note only mentions a very short portion of what mathematicians have said about the subject. But of course, this is only a minimal part of the story. Much more is known, and what it still remains unknown is not only to be known in future, but it will also trigger new questions and, then, new ignorance to fill. So the story has no end, and in fact, like any story of science, it cannot have an end.

The second question is more difficult. A way to begin answering is observing that, even if our four infinities pertain to a single notion, they surely give rise to distinct theories. Distinct, but connected of course, as the entire body of mathematics is. Connection is not to be reason of confusion, however. Even if it there were only a single notion of infinity, this should be studied from different perspectives that complement them, but do not confuse one with another.

But maybe our answer is that the question is not really about the independent existence of one or more infinities, or notions of infinity, but about our human capacity of (finitely) mimic infinity in different ways. How much these ways are complementary becomes then no more a general, philosophical, or methodological question, to be answered by outside of the relevant theories. It rather becomes a question for these theories themselves: a question to be tackled by searching mutual applications, interactions and common fields of study. This makes impossible to even begin answering the question without entering the relevant mathematical theories and working within them.

Acknowledgments

The last two authors gratefully acknowledge the kind invitation to lecture at the Accademia Gioenia, and the very warm welcome they were given. A special thanks also to two anonymous referees for their very fruitful remarks.

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