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Leonardo Fibonacci and Frederick II: An encounter of Islamic mathematics with Europe

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Summary

This note describes a very specific moment in history when Islamic mathematics and European culture came in contact with each other. While such contacts took place over several centuries, this note focuses on Leonardo Fibonacci's experience in North Africa that led him to interact with the scientists at the court of Frederick II, *Stupor Mundi*. This encounter had several important consequences, among which the introduction in the Western world of the Hindo-Arabic numbering system and the beginning of the transition from Arabic to Latin as the language of science and mathematics.

Keywords: Fibonacci; Frederick II; Arab and Islamic mathematics; *Liber Abaci.*

Leonardo Fibonacci e Federico II:
L'incontro della matematica islamica con l'Europa

Riassunto

Questa nota descrive un particolare incontro tra le culture europee e la cultura islamica. Questi incontri, naturalmente, sono avvenuti lungo l'arco di molti secoli, ma in questa nota ci concentreremo sull'esperienza di Leonardo Fibonacci in Africa, esperienza che lo portò ad interagire con gli scienziati della corte di Federico II, *Stupor Mundi*. Queste interazioni ebbero molte importanti conseguenze, tra le quali l'introduzione nel mondo occidentale del sistema di numerazione indo-arabo, e l'inizio della transizione dall'arabo al latino, come linguaggio della scienza e della matematica.

Parole chiave: Fibonacci; Federico II; Matematica araba ed islamica; *Liber Abaci.*

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1. Introduction

This note deals with a moment in the long history of the evolution of the language of mathematics, and in a sense the language of science. There are no doubts that in the antiquity, the language of science and mathematics was Greek. This of course does not mean that other civilizations had not been important contributors (it suffices to think of the Egyptians, the Babylonians, the Indians, the Chinese), but when we think about the great mathematical texts of antiquity, we think of ancient Greece, and the names that jump to our attention are those of Pythagoras, Eudoxus, Euclid, Archimedes, Apollonius, Diophantus, just to limit ourselves to the most famous among them. It is worth noting that Greek remained the language of mathematics even after the political and military power of the Greek cities faded, when the Romans took over as the leading power in the Mediterranean and beyond. We can say that even though it was a Roman solder who killed Archimedes in Syracuse after the successful siege of 212 b.C., Latin did not replace Greek as the language of mathematics.
It was another political, economical, and religious power who broke this tradition and, starting with the 8th century, mathematics began to speak (and to be spoken) in Arabic. This was the natural consequence of the collapse of the Western Roman Empire (476 a.D.), and the progressive weakening of the Eastern Roman Empire in the centuries to follow. These two facts combined with the explosive growth of Islam, not just as a religion, but as a political and cultural force, and by the year 750, under the Ummayyad Caliphate, the entire Southern and most of the Eastern Mediterranean, as well as Persia and Spain, were under Islamic control.

The year 750 is important because it was then that the Ummayyad Caliphate was defeated and replaced by the Abbasid, who moved the capital to Baghdad, and in the next few years the fifth Abbasid Caliph, Harun al-Rashid, built, to house his library, what the seventh Caliph al Ma'mun turned into a public academy, the House of Wisdom, *Al Bayt al Hikmah*. It is not an exaggeration to say that the House of Wisdom was the most important center for the translation movement that took Greek, Syriac, and Indian texts, and translated them into Arabic (Gutas, 1998). Of course, these were not just simple translations. Indeed, the translations became the starting point for a rebirth, a renaissance, of original ideas and scientific progress. When we speak of original ideas being born at the House of Wisdom, we should at a minimum mention the names of al-Khwarizmi, a Persian mathematician working in Baghdad to whom we owe the solution of quadratic equations (Rashed, 2009), but also (a complete list is beyond the scope of this short note) the astronomer Ibn al-Haytam, known in the West as Alhazen (Rashed, 2017), the geometer al-Hasan ibn Muysa ibn Shakir, famous for his very accurate measurement of the circumference of the earth, for a book on the measurement of spherical and plane figures, and for

Kitab al-Hiyal, a book on mechanical devices (Sato, 1985), Omar Khayyam, a major contributor to the study of cubic equations (Rashed, 1999), the astronomer and mathematician al Farghāni, whose work completed an important theorem of Apollonius and gave solid foundations to the utilization of the astrolabe (Lorch, 2005, and Gentili et al., 2020), the great mathematician Abu Ja'far al-Khazin (Rashed, 1984), whose work influenced in significant ways the work of Fibonacci as we will discuss in the next sections, and Abu Kamil (Rashed, 2012), an Egyptian mathematician of whom as well we will speak later on. The result of this intellectual effort, of this investment, was that Arabic became, *de facto*, the language of mathematics and science for many centuries.

The Greco-Roman world and the Islamic world were, of course, in continuous contact during those centuries. Sometimes the contact was through wars (such as the Crusades in the Holy Land), but most frequently the contacts were due to merchants who moved goods across the Mediterranean, from the East to the West, and back. Spices, and raw silk left the East to reach Europe, where clothing and refined textiles were then coming back. Despite the frequent military encounters, the need and desire for trade were more powerful, and as the Swabes and the Normans reconquered Southern Italy, the opportunities for such encounters continued to grow (Bosker et al., 2013).

A crucial moment in this continuous exchange between cultures took place at the end of the twelfth century, when the young son of a Pisan merchant joined his father in Northern Africa and learned first-hand a new way of representing numbers. In section 2.1 of this paper I will briefly summarize how this happened, and what was the immediate consequences of this travel, the publication of a momentous work in the history of European mathematics, Fibonacci's ***Liber Abaci*** and, even more important, what Rashed called (Rashed, 2003) '***le prolongement latin del mathématiques arabes***'. I will then dedicate a short section to this masterpiece, to describe both its intent, and some of its most important results. Section 2.3 of this note is devoted to a description of how, through the interaction with Arab mathematics and the intellectuals at the court of Frederick II, Fibonacci evolved into a theoretical mathematician of very high level. What had begun as a desire to help the Pisan merchants, metamorphosed into an important foray in what would eventually become the Italian mathematical school of algebra that incorporated the enduring legacy of Arab and more generally Islamic mathematics.

2. Results and Discussions

2.1. Fibonacci's travels to Béjaïa

We don't know much about the life of Fibonacci. The little we can gather comes from a short account first translated in (Grimm, 1973), that Fibonacci included in the 1227 edition of *Liber Abaci*, and from the dedications of this and his subsequent works, from which we learn about his relationship with several intellectuals at the court of Frederick II. In a recently published book (Struppa, 2024), the author has built on those scant documents, and has imagined what Fibonacci's life might have been. Not a scholarly book, despite its fairly large set of references, but rather an affectionate portrait of an individual whose life changed mathematics in Europe, as one reads in another recently published work (Catastini et al, 2024), where *Liber Abaci* is discussed in great detail.

What we know, however, is sufficient to give us a glimpse of the crucial years of his life. Fibonacci was born around 1170 in the Maritime Republic of Pisa, to Guglielmo and Alessandra Bonacci (his name 'Fibonacci' simply stands for '*figlio di Bonacci*'). We know that his father was working for the Republic, probably as a custom officer (though the level he had attained in his career is not known). We are also aware of a very important historical event that took place on November 15, 1186, when the Almohadi Caliphate that controlled the Maghreb and much of Northern Africa, reached a commercial agreement with the Republic of Pisa (Ouerfelli, 2015). The 1186 agreement was stipulated between the Maritime Republic of Pisa and the Almohad Caliph Abu Yussuf Yáqub al-Mansur. This treaty specified a list of towns in the Maghreb that Pisa could trade freely with, and specified the nature of the posts that Pisa could establish, posts that were known under the name of *fondouks*, an Arab word that is still found in the Italian expression *fondaco*. This was a remarkable moment, because the battles for the control of the Western Mediterranean had been raging for many years, and had seen victories and reversal of fortunes, involving not only the Almohadi Caliphate, but the various Maritime Republics, chiefly Pisa and Genova.

The 1186 treaty had the very important consequence of allowing Pisa to open a post in the town of Béjaïa, now known as Béjaïa, a coastal town in what is now the Mediterranean coast of Algeria. Guglielmo Bonacci was sent to man this post (probably with a rather high level of responsibility) and he eventually decided to call his son Leonardo to join him. Béjaïa is one of the most ancient cities in the Maghreb, having been established first by Augustus in the first century a.D.; in later years, after the fall of the Roman Empire, Béjaïa became the capital of the Berber dinasty of the Hammadid and finally, in 1152, was conquered by the Almohadi. It is not a surprise, then, that Béjaïa was, at the time, a very important trading post; in part because of his political role, in part because of its strategic location, and finally because of some very specific aspects of its economy. The location on the shores of the Mediterranean and fairly close to the Italian peninsula, made it the perfect place to receive trading goods from Italy and Northern Europe, on their way to other parts of Northern Africa and beyond. At the same time, Béjaïa was also the arrival point for goods coming from the interior of Africa. Finally, Béjaïa enjoyed a very mild climate, and was the center for a significant development of apiculture. This made Béjaïa the capital of the production of beeswax, a good that was central to its economy, and that is still recognized in the French word *bougie*, which means 'candle', or the Italian word *Béjaïa*, which means 'candleholder.' The importance of beeswax for Béjaïa will end up playing a non-irrelevant role in the worldwide popularity of Fibonacci himself, as we will soon see. The reader interested in a more thorough account of society (and mathematics) in Béjaïa at the time of Fibonacci is referred to (Aissani et al., 2003).

While we have no details on what transpired when Leonardo joined his father, it is obvious (from what we read in his later books) that Leonardo was a clever young man, full of intellectual interests, and he certainly learned to read and speak Arabic, well enough, at least, to read the Arab mathematical translations of the Greek classics; this is discussed in the commentaries of Sigler, (1987, 2002) as well as in (Hughes, 2008). It is also not difficult to imagine Leonardo's interest in matters related to commerce, so that when he came into contact with the Indo-Arabic numbering system he must have quickly realized the importance this system may have for the daily activities of a merchant. It is because of this potential, that Fibonacci decided to write the book that eventually ensured his fame throughout the centuries.

2.2. Fibonacci's *Liber Abaci*

Much has been written on this book, that first appeared in 1202, when Fibonacci returned to Pisa from his travels to Béjaïa. Mathematicians have written about the content of this book, and historians have described the impact it had on Western culture, and especially on the development of markets. It would be therefore presumptuous to attempt an analysis of *Liber Abaci* in this short section, and I refer instead the reader to (Catastini et al., 2024), as well as to the relatively recent English translation (Sigler, 2002).

I will however point out that it is with this book, written in Latin, that a third linguistic period begins for mathematics: from now, and of course with some significant exceptions, mathematics will begin being written in Latin, a tradition that will continue until the end of the 18th century, when the national languages began to impose their domain.

Let me offer a very brief (and certainly incomplete) reflection on the nature of this book. To begin with, it is apparent that the book is not written for mathematicians, but for merchants. It is some sort of manual on how we can use the Indo-Arabic digits to represent any number, and how much this method is superior to the Roman based system in place at the time. It could be objected that Fibonacci was not the first to introduce the Indo-Arabic digits to the West, and this is undoubtedly true. In fact, we know that such numerals already appeared in the West in the *Codex Vigilanus* a manuscript dating back to the year 976 and named after the monk Vigila, its illustrator. More important was the role of Pope Sylvester II, who in the last decades of the tenth century, started using these digits and attempted to make them better known throughout Europe. Finally, we are aware of another important twelfth century text, the *Liber Ysagogarum Alchoarismi ad totum quadrivium* (the *Book of the Introductions of al-Khwarizmi to the entire quadrivium*), whose arithmetical part is probably the most important discussion of the Indo-Arabic system before Fibonacci's (the first translation of the *Liber Ysagogarum* is (Allard, 1992), but the reader should also consult (Allard, 1996), in (Rashed, 1996), as well as (Burnett, 2013), where the geometrical aspects of the *Liber Ysagogarum* are studied). All of this is true, but the first successful *manifesto*, so to speak, was Fibonacci's *Liber Abaci*, where the intent is clearly to represent the importance and advantages of using a positional system. Reading *Liber Abaci* is an exciting and revealing experience. The entirety of its first part can be seen as a propaganda for this new way of counting and representing numbers. Fibonacci provides a wealth of practical advice, including an excursion on indigitization (the use of fingers to calculate with large numbers), that were intended to convince the reader of the utility of this new system (Fig. 1).

Fig. 1. Excursion on indigitization from Fibonacci's *Liber Abaci*.

Among the main advantages, the possibility of writing arbitrary large numbers, something that remained a challenge for the Roman numerical system and that had been one of the motivations for Archimedes' *The Sand Reckoner*, whose standard translation is (Heath, 1897), but also the ease with which one can add, subtract, multiply and divide numbers, and the simplicity with which one can check the results of such operations, without having to do the calculations multiple times. That the target for *Liber Abaci* was the merchant is also plainly visible in the examples that Fibonacci uses throughout the book. One entire section, for example, is devoted to the way in which different investors in a company should be rewarded on the basis of their investment (simple proportion problems, from our perspective, but not obvious to the medieval merchant). Another section is devoted to a practical method to allow merchants to barter different goods once their costs are known in some common currency. This is a problem that requires a double proportion, but what is remarkable about Fibonacci's approach is not the way in which he finds the solution, but rather how he devises a mnemonic that the merchant can utilize without needing to understand the mathematics behind it. Yet another section deals with the question on how to mint coins of a certain value by appropriately using different alloys. It is in the course of discussing this problem, however, that a different aspect of Fibonacci begins to emerge, and the mathematician replaces the teacher and the merchant. Indeed, Fibonacci now considers a series of intriguing problems that he had first encountered in Béjaïa (and that the Arabs had themselves inherited from the Indians), and that amount to finding integer solutions of systems of equations with integral coefficients (diophantine problems, in current parlance). Fibonacci understands how these problems can be interpreted as coin minting problems (he thus establishes an isomorphism between these apparently different problems) and attempts to offer a general method to solve them. Reading this second portion of *Liber Abaci* is an exploration in the development of new mathematical ideas, some of which will be at the basis of Fibonacci's subsequent work. More important from our point of view, this second portion of the book clearly shows its relationship with Arabic mathematics. Not only many of the problems are an echo of questions Fibonacci was probably studying in Béjaïa, but some of the methods that he develops, can be directly traced back to mathematicians such as Abu Kamil. An example of this is the treatment that Fibonacci gives of the method of false position (used to solve first degree equations of the form $ax = b$) and, more importantly, of the method of double false position, that he calls *elchatayn*, a corruption of the Arabic expression *hisab al-khata'ayn*, that roughly translates to 'calculating through two errors', and that is used to solve first degree equations of the form $ax + b = 0$. The impact that Abu Kamil had on Fibonacci is well known, and a reader interested in the details is referred to (Karpinski, 1914) where the author examines in detail manuscript 7377*A* maintained at the Bibliotèque Nationale in Paris, and identifies the passages that Fibonacci, in the

Liber Abaci, elaborates, as well as to the exhaustive (Rashed, 2012). This transformation from the desire to help merchants to the desire of understanding fundamental properties of numbers is a significant source of interest in *Liber Abaci*, and will become even more evident once Fibonacci begins to interact with the court of Frederick II.

2.3. The encounter of Fibonacci with the court of Frederick II

I have, so far, only spoken of Fibonacci: his birth in Pisa, his travel to Béjaïa, the encounter with the Hindo-Arabic system, and his first major work, the *Liber Abaci*. But as we read the only version of the *Liber Abaci* that is available to us, a version that first appeared in Pisa in 1227 and that reached us through the work of Boncompagni (1857), we finally encounter another major character in this story, a character that while not a mathematician, or a scientist in the traditional sense of the world, still played a crucial role in the transmission of Arab ideas to the West. I am referring to the greatest western medieval emperor, Frederick II (known as *Stupor Mundi*, or 'The wonder of the world'). As I said, all we know about Fibonacci's life is in the introduction he wrote to this version of *Liber Abaci*, but equally instructive is the dedication of this book, which I report here verbatim. Fibonacci writes: '*Scripsistis mihi domine mi magister Michael Scotte, summe philosophe, ut librum de numer, quem dudum composui, vobis transcriberem; unde vestrae obsecundans postulation, ipsum subtiliori perscrutans indagine ad vestrum honorem et aliorum multorum utilitatem correxi. In cuius correctione quedam necessaria addidj, et quedam superfula resecaui. In quo plenam numerorum doctrinam adidij, iuxta modum indorum, quem modum in ipsa scientia prestantiorem elegi*'.[†]

The reference to Michael Scot (or Michael Scotus as he would have been known at the court) is telling because he was the court astrologer for Frederick II. Scotus was Scottish, as his name indicates, born around 1175 either in the town of Fib, now known as Fife, or in the town of Borders (both towns claim him as their own descendant) and one of the most powerful men at the court of the emperor. His scholarly career led him to all the great medieval universities, Oxford, Paris, and Bologna, before he landed at the court of Frederick II. He was well known for his ability to interrogate the stars and make prediction on human events (Andriani, 2022), and the author of many important texts that combined scientific discovery with theological research and astrology. It is because of his role as magician, negromante, and astrologer that Dante places him in the fourth *bolgia* of the eight circle of Hell, the so-called '*malebolge*', and

† In (Sigler, 2002) this is translated as *My Lord and teacher, greatest philosopher, Michael Scotus, you have asked me to copy for you the book on numbers that I wrote a while ago. And so, to acquiesce to your request, and after carefully reviewing it, I have corrected it in your honor and to make it useful to many others. During the correction I have added something that was necessary, and removed something that was unnecessary. In this book I have described the entire theory of numbers according to the Indian method, that I have chosen as the most efficient in this science*.

describes him, in Canto XX, verses 115-117, as '*Quell'altro che ne' fianchi è così poco, Michele Scotto fu, che veramente de le magiche frode seppe 'l gioco*.'‡

Famously, as reported in (Brown, 1897), the legend says that Scotus predicted that Federico would die *ad portas ferreas*, namely near some iron gates, in a town named after the goddess Flora. Federico seemed to have understood this to indicate the city of Florence, possibly at the church of Santo Stefano (where such gates were said to exist) and avoided going to that city. Unfortunately, in 1250, he ended up falling sick at the town of Fiorentino (now Castel Fiorentino) in the Puglia region, where he passed away, apparently in confirmation of Scotus' prophecy. It is also alleged that Scotus predicted his own death. According to this prediction (Benvenuti, 1887) Scotus believed he would be killed by a stone falling on his head. He therefore always wore a metal skullcap to protect his head. One day, however, entering a church during the Corpus Domini holiday, he removed the protection, and a stone fell from the ceiling, causing a wound that ended up killing him. One more legend on Scotus will explain the trust that his benefactor Frederick II had in him. Apparently, Federico asked him to measure the distance of the heavens from his palace. Scotus, it is not clear how, came up with some number. Taking advantage of a trip that took Scotus away from Palermo, where the palace was located, Federico lifted the pavement of the palace and, when Scotus returned from the trip, he asked him the same question. The story goes that Scotus modified his response exactly of the amount of which Federico had lifted the pavement, thus earning the eternal respect and trust of the emperor (a skeptic might suggest that Scotus had a friend at the court who informed him of what had happened.).

Going back to the dedication that Fibonacci made of the second version of *Liber Abaci* to Michael Scotus, we should note that one of the new questions that, apparently, Scotus had suggested to Fibonacci is the one that led him to the introduction of the so-called Fibonacci sequence. Let us take a moment to both discuss the Fibonacci sequence, the problem that Scotus had posed, and finally a possible alternative explanation.

The Fibonacci sequence begins with the numbers 0 and 1, and it continues by adding the last two numbers of the sequence to produce the next term. It is what mathematicians call a 'recursive sequence' that is fully specified by setting $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for all $n \geq 2$; we say that F_n is the n-th Fibonacci number. So, from $F_0 = 0$ and $F_1 = 1$ we obtain $F_2 = 1$. Now the last two numbers are $F_1 = 1$ and $F_2 = 1$, and so the next term will be $F_3 = 2$, and then $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, and so on and so forth. This sequence is very well known among amateur mathematicians, and among the public at large, in part because it seems to

‡ *The next, who is so slender in the flanks, Was Michael Scott, who of a verity Of magical illusions knew the game*, in the translation of (Longfellow, 1867).

appear very frequently in nature. For example, the number of petals on many flowers is a number in the Fibonacci sequence, just like the number of arms on most starfish is a Fibonacci number. The way Fibonacci introduces such a sequence in his book is rather artificial and seems to have been suggested by Michael Scotus. Here is the question: *a pair of rabbits, one male and one female, are put in a courtyard. These rabbits mate after one month and so at the end of the second month the female produces another pair of rabbits (still a male and a female). We will further assume that these rabbits never die, and that every pair produces one new pair every month after the second month of life. It is asked how many pairs there will be after twelve months.* We leave it to the reader to analyze the problem, but it is not too difficult to see that after one month there is one pair of rabbits, after two months there is still only one pair of rabbits, but after three months there are two pairs of rabbits, after four months three pairs of rabbits, and so on and so forth, so that after n months, the number of rabbits is exactly F_n.

This is what we learn from Fibonacci's second edition of *Liber Abaci*, but the problem that is used is, to say the least, artificial. The description of how rabbits reproduce is not accurate, and the conditions that Fibonacci stipulates in order for the problem to yield his famous sequence appear quite unnatural; the rabbits are supposed to reproduce at regular intervals, and we need them to give birth to exactly one male and one female, and this has to happen over and over again. It is not very likely that this is the real story that originates the famous sequence. There is, however, a different interpretation that has been advanced in (Scott and Marketos, 2014). It is interesting enough that is worth describing. We already discussed the travels of Fibonacci to Béjaïa, and we mentioned the importance that apiculture had for Béjaïa's commerce. It is quite possible to imagine that Fibonacci's interest in the sequence that will take his name was derived by the singular reproduction process among bees. Indeed, as we know, there are three kinds of bees in a hive. The female workers, the drones, and the queen bee. The queen bee is the only bee who lays eggs. If an egg is not fertilized, after being laid, it will hatch and turn into a drone (a male bee). Drones' job is to fertilize the eggs laid by the queen been, and those eggs who are fertilized will eventually hatch as female workers (all of which, with one exception, will be infertile). Among all female bees, one is fed a special meal, the so-called royal jelly, and that special nourishment turns it into the fertile queen bee. Let us therefore try to count the ancestors of a drone. A drone only has a mother (the queen bee) since it comes from an unfertilized egg. The queen bee, however, has two parents, the previous queen bee from whose egg she originated, and the drone that fertilized the egg. Thus, the original drone has one parent, but two grandparents. The two grandparents are a queen bee (who has two parents) and a drone (only one mother). This shows that our original drone has three great-grandparents, two of which are queen bees, and one is a drone. So, at the next step, we will find five great great grandparents. It is now obvious to see that the number of ancestors of a drone, at each generation, is given by the appropriate Fibonacci number. This explanation for how Fibonacci came out with the idea of the sequence seems much more plausible, and it has the advantage of being rooted into his sojourn in Béjaïa. Figure 2 is taken from the most complete website we ever saw regarding bees, due to David Kushman, and that can be retrieved at http://www.dave-cushman.net/bee/fibonacci.html

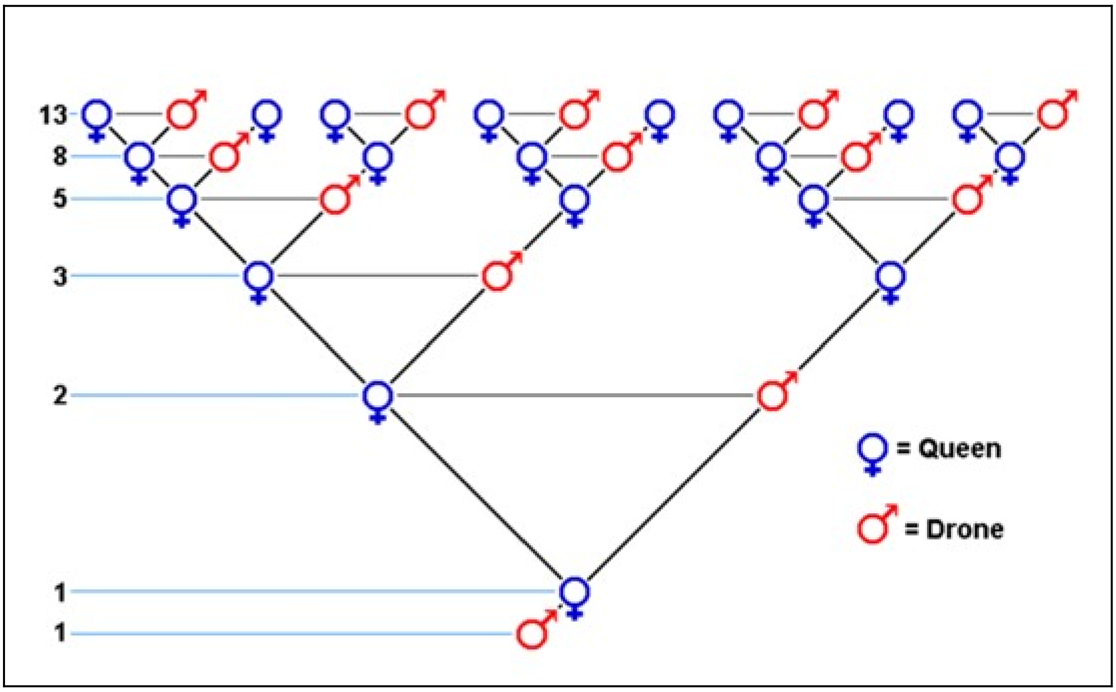

Fig. 2.

There are however several other scientists, in addition to Scotus, serving under Frederick II, who impacted the work of Fibonacci, and developed further connections with the mathematics of Arab Northern Africa. Among these, I will briefly discuss three names: Master Dominicus, Master Ioannis, and Master Theodorus. Let us begin with Master Dominicus. While it is not completely clear who Fibonacci is referring to with this name, the current belief identifies him with Dominicus Hispanus, one of the great translators from the Toledo School, the school that, in the years immediately following the *Reconquista* of Toledo carried out by Alfonso VI the Brave, in the year 1085, picked up the same idea that had animated Baghdad's House of Wisdom, and began a systematic translation of all Arabic texts, thus providing further energy to the process by which Latin began to replace Arabic as the language of science. If this is indeed the Master Dominicus that exchanged mathematical ideas and challenges with Fibonacci, we know that he was a very prolific translator, and that he was the most important stimulus behind the second great book of Fibonacci, *De Practica Geometriae*, which Fibonacci completed in the years 1220-23. We know of his impact because Fibonacci himself, in the introductory note to *De Practica Geometriae*, writes: *Rogasti Amice et Reuerende Magister, ut tibi librum in pratica geometriae conscriberem; Igitur amicitia tua coactus tuis precibus condescendens, opus iam dudum inceptum taliter tui gratia edidi... ... Ad hec igitur secundum ingenij mei capacitatem perficienda, tue correctionis aggressus fiducia, hoc opus curaui tuo magisterio destinare, ut que in eo fuerint emendanda, tua sapientia corrigantur.*[§] It should be pointed out that, once again, the purpose of this book is an educational one. Just like Fibonacci attempted to use *Liber Abaci* to instruct the merchants in the use of these new digits, so Fibonacci is now trying to provide practical instruction to the surveyors, the urban planners, and the magistrates who needed to apportion disputed land. The book (at least its first part) is essentially a manual to calculate areas especially in difficult situations when such areas located on the sides of mountains. Despite this practical intent (evident even in the title), Fibonacci does not abandon his interest in more arithmetical questions and spends considerable time in discussing a method to calculate square and cubic roots, thus improving on what he had done in *Liber Abaci*. The importance of Master Dominicus, however, is not restricted to his role in inspiring the writing of *De Practica Geometriae* but also (if not even more important) in organizing and coordinating the visit that Fibonacci paid to Emperor Frederick II. Indeed, in 1224, Master Dominicus proposed that Fibonacci could be invited to meet the emperor on the occasion of an imperial trip to Northern Italy. The emperor, presumably after traveling through Cosenza (where he had participated to the consecration of the Duomo on January 30, 1222) and through Napoli (where on June 5, 1224 he had founded the university that even today carries his name) reached Pisa sometime in 1225, and there he welcomed Fibonacci.

It was during this meeting, that another important figure emerges, Master Johannis of Palermo. As always, it is not easy to identify conclusively this figure. According to (Kantorowicz, 1927), there were two scholars under this name, one a notary at the imperial court, and the other a mathematician, who was sent in a mission to Tunis in 1240, and who is the author, at least, of the Latin translation *De duabus lineis* of an Arabic work on the hyperbola. We learn about this encounter from the introduction that Fibonacci writes to his short booklet *Flos* where he dedicates his work to Frederick II with these words: *Cum coram maiestate vestra, gloriosissime princeps, Federice, magister Johannes panormitanus, phylosophus vester, mecum multa de numeris contulisset, inter que duas questiones, que non minus ad geometriam quam ad numerum pertinent, propusuit. Quarum prima fuit*

§ In (Hughes, 2008) this introduction is translated as *Dominicus, friend and reverend teacher! You asked me to write a book on practical geometry. Stimulated by your friendship, and yielding to your request, I rewrote for you a treaty that I had already begun...so that others might have a complete treatise...To do so at the best of my ability, and certain of your ability to make the necessary modifications, I have prepared this work for your examination.*

ut inveniretur quadratus numerus aliqus, cui addito vel diminuto quinario numero, egrediatur quadratus numerus.[**] This question must have been well known to Fibonacci, who might have already encountered it in his study of Abu Ja'far al-Khazin, and it is an intriguing one. How can we find a square number x^2 such that both x^2-5 and x^2+5 are squares. The difficulty of this problem lies in the fact that there is no integer solution to this question, and the answer that Fibonacci gives seems almost impossible to find. Indeed Fibonacci shows that $\frac{41}{12}$ is a solution to Master Johannis' question, since the square of $\frac{41}{12}$ added to 5 gives the square of $\frac{49}{12}$ while when 5 is subtracted from the square of $\frac{41}{12}$ one obtains the square of $\frac{31}{12}$. Fibonacci gives the solution in *Flos*, where, after reporting the question, he continues *"... ut eidem magistro Iohanni retuli, inueni esse hunc numerum, vndecim et duas tertias et centesimam quadragesimam quartam unius. Cuius numeri radix est ternarius et quarta et V I*[a]*, unius"*, which translates as *"... as I told Master Johannis, I found this number to be eleven and two thirds and one over one-hundred-forty-four. Whose root is three and a quarter and a sixth".* Using numbers rather than words this means that the solution to the problem posed by Master Johannis is $11+\frac{2}{3}+\frac{1}{144}=\frac{1681}{144}$, whose root is indeed $3+\frac{1}{4}+\frac{1}{6}=\frac{41}{12}$. The reader is referred to (Kouteynikoff, 2020) for a discussion on how this question may have reached Master Johannis.

It is in this situation that Fibonacci shows, however, that he had become a real mathematician, as he counters the question of Master Johannis with a question that only a mathematician could ask. Namely he wants to know whether the fact that he was able to find a solution is due to some special property of the number 5. More specifically, Fibonacci asks for which integers N, one can solve the following problem: find a square such that when one adds or subtracts N, one still gets a square. It is telling of Fibonacci's growth as a mathematician the fact that he realized that the solvability of the problem depended on the choice of N, and that those numbers for which the problem had solution were somewhat special and worthy of independent investigation. He called a number N for which the problem could be solved a *congruum* because it made two different

** This book, as far as I know, has not yet been translated in English and this dedication can be translated as: "*In front of Your Majesty, oh glorious prince Federico, Master Johannis of Palermo, philosopher of your court, asked me many questions on numbers, and in particular two question that are connected to geometry, no less than to numbers. The first such question was to find a square, such that when we add or subtract five, we will still find a square.*"

squares congruent.†† This story further highlights the nature of the interaction between Arabic and European mathematics. Ideas that had germinated among Arabic mathematicians are identified as interesting by different European mathematicians (in this case Master Johannes as well as Fibonacci) and they are further developed to expand their interest and importance, a perfect example of Rashed's notion of *prolongement*. This exchange is particularly important because Fibonacci will find in it the impulse to write what many consider his masterpiece, the *Liber Quadratorum*, whose importance is discussed in the commentaries in (Sigler, 1987) as well as in (Bussotti 2008, 2015). That this question was the impulse for the Liber Quadratorum is clear from its introduction, where Fibonacci writes "*Cum Magister Dominicus pedibus celsitudinis vestre, princips gloriosissime domine F., me Pisis duceret presentandum, occurrens Magister Iohannes panormitanus...iet opus incepi ad vestrum honorem condere infrascriptum. quod vocari librum volui quadratorum, veniam postulans patienter, si quid in eodem plus vel minus iusto vel necessario contenitur, com omnium habere memorial, et in nullo peccare, sit divinitatis potius quam humanitatis, et nemo sit vitio carens, et undique circumspectus*" which is translated in (Sigler, 1987) as "*After Dominicus brought me to Pisa, at the feet of Your celestial majesty, oh most glorious prince, Lord Federico, I met Master Johannis of Palermo...I then decided to deal with these questions and began to write, in Your honor, this book that I will call Liber Quadratorum. I ask for Your indulgence if some passages contain more or less of what is necessary, since to know everything and never err is divine, and not human, and nobody is exempt from mistakes.*"

The encounter with Emperor Frederick II concludes with another challenge to Fibonacci, this time a tougher one, and posed to him by Master Theodorus. Before stating the question that Theodorus poses, we note that he was a scholar originary of Antiochia, and with whom Fibonacci will entertain important correspondence, including his famous, and undated, *Epistola suprascripti Leonardi ad Magistrum Theodorum phylosophum domini Imperatoris*, usually referred to as the *Mathematical letter of Fibonacci to Master Theodorus*, and that has reached us thanks to (Boncompagni, 1857), where it is printed for the first time. In this exchange we find a diophantine problem that Theodorus had learned about from the Arabs, who in turn had learned about it from the Indians. The problem is simple to state; a man buys 30 birds of three different kinds: partridges (which cost 3 *denari* each, pigeons (which cost 2 *denari* each), and sparrows (which cost 1 *denaro* each). The question is how many birds he buys of each kind. In his letter to Master Theodorus, Fibonacci solves this and many variations of this problem, but what is interesting is that he does so by showing an analogy with the problem of making a coin with a certain alloy of different metals, a problem that

†† The theory of congruent numbers that is connected with this question, see more details in (Kouteynikoff, 2020), is still of modern interest, as seen for example in (Sebbar, 2024).

had a relevance in the financial system of medieval Italy, where different cities were using different coins (often with the same name) and different alloys of metals.

But let us now get back to the encounter with Frederick II, when Fibonacci is posed a much harder problem by Master Theodorus. Specifically, he is asked to find three numbers such that, added to the square of the first, give a square. This square, added to the square of the second number gives again a square. And finally this square, added to the square of the third number, must give one more square. In modern notation this problem consists in solving (with integral solutions) the following system of equations:

$$x + y + z + x^2 = w^2$$
$$x + y + z + x^2 + y^2 = q^2$$
$$x + y + z + x^2 + y^2 + z^2 = p^2.$$

Unlike the question posed by Johannes of Palermo, this question was too hard for Fibonacci to answer immediately, but he later discussed it in the *Liber Quadratorum*, and for this we refer the reader once again to (Sigler, 1987).

As we conclude this brief note, we observe how the encounter of Fibonacci with the court of Frederick II helped planting the seeds for what would eventually become the Italian school of algebra. It would take a while longer for these seeds to give their greatest fruits, through the theory of solutions of polynomial equations of degrees three and four. The story of how that happened is of great interest and would lead us to the names of Bombelli, Scipione del Ferro, and Tartaglia among others, culminating with the work of Girolamo Cardano. We limit ourselves to pointing out how the role played by Fibonacci and by al-Khwarizmi's *al-Kitāb alMukhtas.ar fīsāb al-Jabr wal-Muqābalah*[‡‡] was fully recognized by Cardano in his *Ars Magna* written in 1545,[§§] and where he develops a full theory for the equations of degree three and four. Indeed, at the very beginning of his work, in Chapter I, Cardano writes: *Haec Ars olim à Mahomete, Mosis Arabis filio initium sumpsit. Et enim huius rei locuples testis Leonartus Pisanus.*[***] This quote, from the very beginning of Cardano's book, not

‡‡ A translation of this title would be *The compendious book on calculation by completion and balancing*, and the reader will easily identify in the Arab title the word *al-Jabr* that means completion, but from which our own word *algebra* comes from. Al-Khwarizmi's book was written around the year 820. A translation with the Arabic text as well, can be found in (Rashed, 2009).

§§ Girolamo Cardano actually entitled his work *Artis Magnae, Sive de Regulis Algebraicis Liber Unus*, (Cardano, 1545), which we can translate as *Book One, about the Great Art, or The Rules of Algebra.*

*** As translated in (Witner, 1968), Cardano says: "This art originated with Mahomet the son of Moses the Arab. Leonardo of Pisa is a trustworthy source for this statement"; the reference to al-Khwarizmi as "Mahomet the son of Moses the Arab" is due to the full name of al-Khwarizmi, that is, Muhammad ibn Musa al-Khwarizmi, which means Muhammad son of Moses from (the region of) Khwarazm. In a footnote to the translation, Witner observes that Cardano later pointed out that his reference is due to a only acknowledges the role of al-Khwarizmi, but also Fibonacci's role in creating the bridge.

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