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Sharp a priori estimates for some nonlinear elliptic problems †

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Dedicated to the friend Mario Marino on the occasion of his 70-th birthday.

Summary

We consider homogeneous Dirichlet problems whose prototype is

$$-\Delta_p u = h|\nabla u|^q + f \quad \text{in } \Omega , \qquad u = 0 \quad \text{on } \partial\Omega .$$

We give existence results focusing our attention on the sharpness of the conditions on the known term f. All the estimates are the result of comparison results obtained via symmetrization methods.

Key words: *A priori estimates, existence results, nonlinear elliptic equations, rearrangements, symmetrization methods.*

Riassunto

Si considerano problemi di Dirichlet del tipo

$$-\Delta_p u = h|\nabla u|^q + f \quad \text{in } \Omega , \qquad u = 0 \quad \text{su } \partial\Omega .$$

Alcuni risultati di esistenza vengono illustrati. Particolare attenzione è rivolta alla ottimalità delle condizioni da imporre sul termine noto f. I risultati sono conseguenza di alcune stime fini di confronto ottenute con metodi di simmetrizzazione.

Parole chiave: *Maggiorazioni a priori, risultati di esistenza, equazioni ellittiche non lineari, riordinamenti, metodi di simmetrizzazione.*

1 Introduction

We consider a problem in the form

$$\begin{cases} -\text{div}\,(A(x,u,\nabla u)) = H(x,\nabla u) + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

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where Ω is an open bounded subset of $\mathbb{R}^N$,

$$A \,:\, (x,s,z) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow A(x,s,z) \in \mathbb{R}^N$$

$$H \,:\, (x,z) \in \Omega \times \mathbb{R}^N \longrightarrow H(x,z) \in \mathbb{R}$$

are Carathéodory functions which satisfy the ellipticity condition

$$A(x,s,z)\cdot z \geq |z|^p \tag{2}$$

with $1 < p < N$, the monotonicity condition

$$(A(x,s,z) - A(x,s,z'))\cdot(z - z') > 0\,, \quad z \neq z' \tag{3}$$

and the growth conditions

$$|A(x,s,z)| \leq a_0|z|^{p-1} + a_1|s|^{p-1} + a_2 \tag{4}$$

$$|H(x,z)| \leq h\,|z|^q \tag{5}$$

with $p-1 \leq q \leq p$.

The results we have in mind must show the most general conditions on the term f which guarantee the existence of a solution to problem (1). These conditions can be seen as a sort of thresholds beyond which Dirichlet problem (1) could miss some basic properties as existence or uniqueness in the small.

Obviously the largest functional space where we can put f is $L^1(\Omega)$. However, in this case the usual notion of weak solution does not work. So it is necessary to introduce an ad hoc definition. This question goes back to the paper [1] by G.Stampacchia where a linear elliptic operator is considered. The first extension to nonlinear elliptic operators is due to L.Boccardo & Th.Gallouët [2]. Subsequently, further procedures to provide admissible extensions of the concept of solution have been introduced (see [3], [4], [5], [6]). Each one of them exhibits at the first stage appropriate a priori estimates. Well, our aim is to get a priori bounds which can be considered sharp in some sense. The strategy consists in looking for a problem whose solution is the largest possible in the sense of the rearrangements. That's the reason why we call it the "worst problem". It is radially symmetric; so we can reduce the proofs to unidimensional cases.

Let us shortly recall the definition of rearrangement.

If u is a measurable function defined in Ω and

$$\mu\,(t) = |\{x \in \Omega : |u\,(x)| \geq t\}|$$

is the distribution function of u, then the decreasing rearrangement of u is

$$u^*\,(s) = \sup\,\{t \geq 0 \,:\, \mu\,(t) > s\}\ .$$

Moreover, if ω_N is the measure of the unit ball of $\mathbb{R}^N$ and $\Omega^\#$ is the ball of $\mathbb{R}^N$ centered at the origin with the same measure as Ω

$$u^\#\,(x) = u^*(\omega_N\,|x|^N)$$

denotes the spherically decreasing rearrangement of u.

Rearranging turned out an effective method to obtain sharp a priori estimates since the celebrated paper by G.Talenti [7] (see [8] for the extension to nonlinear operators). The procedure has been adapted to linear elliptic operators with first lower order terms (see [9], [10], [11]) and to nonlinear elliptic operators like that in (1) with $q = p-1$ (see [12], [13], [14], [15] and, also, [16] for a different approach). So, if $f \in L^1$, the question of the existence of a solution to (1) can be considered quite clear at least when $q = p-1$.

On the contrary, when $q > p-1$ the question is more intricate.
Indeed L^1 is not the right space for f when

$$q \geq \frac{N(p-1)}{N-1} = q_l\,. \tag{6}$$

Besides, a smallness condition on the norm of f appears as pointed out in various papers (see [17]-[26] for example).

If

$$q < q_l \tag{7}$$

we can observe this phenomenon when the datum f shrinks in the origin.

Let us consider the radially symmetric problem

$$\begin{cases} -\Delta_p v = h\,|\nabla v|^q + K\,\delta_0 & \text{in } \Omega^\sharp \\ \\ v = 0 & \text{on } \partial\Omega^\sharp \end{cases} \tag{8}$$

where δ_0 is the Dirac measure. A solution exists if and only if $K < K_1$ where

$$K_1 = \frac{N\omega_N^{\frac{q}{N(q-p+1)}}}{|\Omega|^{\frac{q-N(q-p+1)}{N(q-p+1)}}}\left(\frac{q-N(q-p+1)}{h(q-p+1)}\right)^{\frac{p-1}{q-p+1}}\,. \tag{9}$$

Problem (8) occupies a special place in this context. It can be seen as the "worst problem". Indeed we can prove that the rearrangement $u^\sharp$ of the solution u to (1) is less than the solution v to (8) if the L^1-norm of f is less than K_1.

Our first aim is to give an answer to the following question: does problem (1) have a solution if

$$\|f\|_{L^1} < K_1 \tag{10}$$

when q satisfies condition (7)?

An analogous question can be expressed when $q \geq q_l$. However, in this cases L^1 is not the right space for the term f. Moreover we observe that, as q increases, the right functional space for f contracts.

The note is organized as follows. Section 2 provides the basic estimates for the rearrangement of the solution to (1). These are the starting points to lead, first, to a priori bounds, then, to the existence of a solution. We refer to the notion of solution introduced in [5] and called SOLA, acronym of Solution Obtained as a Limit of Approximations. It is obtained as limit of a sequence $\{u_n\}$ of weak solutions to problems as (1) with H replaced by its truncations at the levels n and f replaced by data $f_n \in L^\infty$ converging in some sense to f. In Section 3 we outline these results; for the proofs we refer the reader to [27].

2 Pointwise comparison results

As remarked in the Introduction the symmetrization techniques allow to reduce the proof of a priori estimates for solutions to Dirichlet problems to that for solutions to simpler problems. Actually they are unidimensional thanks to the radial symmetry of the data.

In the present section we give pointwise estimates whose complete proofs are in [27]. As pointed out they depend on the interval where we put q.

We begin by considering the case (7).

Theorem 1 *Let us suppose that* (2), (5) *and* (7) *hold true. If* $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *is a weak solution to problem* (1) *with* f *bounded and* (10) *is satisfied with* K_1 *given by* (9) *then*

$$u^*(s) \le v(s) \tag{11}$$

where

$$v(s) = \frac{\|f\|_{L^1}^{\frac{1}{p-1}}}{(N\omega_N^{1/N})^{\frac{p}{p-1}}} \int_s^{|\Omega|} \frac{\sigma^{-\frac{p(N-1)}{N(p-1)}}}{\left[1 - \left(\frac{\|f\|_{L^1}}{K_1}\right)^{\frac{q-p+1}{p-1}} \left(\frac{\sigma}{|\Omega|}\right)^{1-\frac{q(N-1)}{N(p-1)}}\right]^{\frac{1}{q-p+1}}} d\sigma$$

is the decreasing rearrangement of the solution to problem (8) *with* $K = \|f\|_{L^1}$.

Sketch of the proof. We recall the following inequality for the rearrangement u^* of u proved in [19] (see also [11])

$$\begin{aligned} &(N\omega_N^{1/N})^{\frac{p}{p-1}} s^{\frac{p(N-1)}{N(p-1)}} [(-u^*)'(s)] \\ &\le \left[\int_0^s f^*(\sigma) \exp\left(h(N\omega_N^{1/N})^{q-p} \int_\sigma^s \frac{[(-u^*)'(r)]^{1-p+q}}{r^{\frac{(p-q)(N-1)}{N}}} dr\right) d\sigma\right]^{\frac{1}{p-1}} . \end{aligned} \tag{12}$$

Putting

$$U(s) = \int_0^s f^*(\sigma) \exp\left(h(N\omega_N^{1/N})^{q-p} \int_\sigma^s \frac{[(-u^*)'(r)]^{q-p+1}}{r^{\frac{(p-q)(N-1)}{N}}} dr\right) d\sigma$$

(12) turns into the problem

$$\begin{cases} U'(s) \le f^*(s) + \dfrac{h}{(N\omega_N^{1/N})^{\frac{q}{p-1}}} \dfrac{U^{\frac{q}{p-1}}}{s^{\frac{q(N-1)}{N(p-1)}}} \\ \\ U(0) = 0 . \end{cases}$$

On the other hand, a direct computation gives

$$(N\omega_N^{1/N})^{\frac{p}{p-1}} s^{\frac{p(N-1)}{N(p-1)}} [(-v)'(s)] = (V(s))^{\frac{1}{p-1}} \tag{13}$$

where

$$V(s) = \frac{\|f\|_{L^1}}{\left[1 - \frac{N(q-p+1)h}{q-N(q-p+1)} \frac{\|f\|_{L^1}^{\frac{q-p+1}{p-1}}}{(N\omega_N^{1/N})^{\frac{q}{p-1}}} \left(\frac{s}{|\Omega|}\right)^{1-\frac{q(N-1)}{N(p-1)}}\right]^{\frac{p-1}{q-p+1}}}$$

solves the problem

$$\begin{cases} V'(s) = \dfrac{h}{(N\omega_N^{1/N})^{\frac{q}{p-1}}} \dfrac{V^{\frac{q}{p-1}}}{s^{\frac{q(N-1)}{N(p-1)}}} \\ \\ V(0) = \|f\|_{L^1} . \end{cases} \tag{14}$$

Here we observe that $V(s)$ and $v(s)$ are well defined and satisfy (13) and (14) because of the smallness assumption (10).

By standard arguments we can deduce that

$$U(s) \le V(s)$$

from which, by using (12) and (13), we get

$$(-u^*)'(s) \le (-v)'(s)$$

and then (11). □

To throw light on the case $q \ge q_l$ we dwell upon the limit case

$$q = q_l\,. \tag{15}$$

A pointwise estimate for the solution to problem (1) holds true also in this case. The main difference concerns the summability of f. Indeed a stronger condition has to be required: f belongs to an intermediate space between L^1 and the Lebesgue spaces L^m with $m > 1$. A "smallness" assumption needs too.

Theorem 2 *Let us suppose that* (2), (5) *and* (15) *hold true. If $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a weak solution to problem* (1) *and f is a bounded function such that*

$$\int_0^{|\Omega|} f^*(s)\,\log^{N-1}(\mathcal{M}/s)\,ds < K_2\,, \tag{16}$$

with $\mathcal{M} > |\Omega|$ and

$$K_2 = \frac{\omega_N (N-1)^{N-1} N^N}{h^{N-1}}\,,$$

then

$$u^*(s) \le w(s) \tag{17}$$

where

$$w(s) = \frac{K_2^{\frac{1}{p-1}}}{(N\omega_N^{1/N})^{\frac{p}{p-1}}} \int_s^{|\Omega|} \frac{1}{\sigma^{\frac{p(N-1)}{N(p-1)}} \log^{\frac{N-1}{p-1}}(\mathcal{M}/\sigma)}\,d\sigma\,.$$

A few comments on condition (16) are in order. It is sharp in the following sense. If R_Ω denotes the radius of $\Omega^\sharp$ and $R > R_\Omega$ the functions

$$u_R(r) = (N-1)^{N-1} \int_r^{R_\Omega} \frac{dt}{t^{N-1}\left[\log\left(\frac{R}{t}\right)\right]^{N-1}} \tag{18}$$

satisfy in $\Omega^\sharp - \{0\}$ the homogeneous equation

$$-\Delta u - |\nabla u|^{\frac{N}{N-1}} = 0\,. \tag{19}$$

Now we define the following regular functions

$$u_\varepsilon(r) = \begin{cases} u_R(r) & \text{if } r > \varepsilon \\ ar^2 + b & \text{if } r \le \varepsilon \end{cases} \tag{20}$$

where

$$a = \frac{(N-1)^{N-1}}{2\varepsilon^N \log^{N-1}\left(\frac{R}{\varepsilon}\right)}\,.$$

Functions (20) solve the Dirichlet problems

$$\begin{cases} -\Delta u_\varepsilon = |\nabla u_\varepsilon|^{\frac{N}{N-1}} + f_\varepsilon & \text{in } \Omega^\sharp \\ \\ u_\varepsilon = 0 & \text{on } \partial\Omega^\sharp \end{cases}$$

where

$$f_\varepsilon(r) = \begin{cases} 0 & \text{if } r > \varepsilon \\ \\ \dfrac{N(N-1)^{N-1}}{\varepsilon^N \log^{N-1}\left(\frac{R}{\varepsilon}\right)} - \dfrac{(N-1)^N r^{\frac{N}{N-1}}}{\varepsilon^{\frac{N^2}{N-1}} \log^N\left(\frac{R}{\varepsilon}\right)} & \text{if } r \leq \varepsilon\,. \end{cases}$$

It is easy to verify that

$$\lim_{\varepsilon\to 0} \int_{\Omega^\sharp} f_\varepsilon(r) \log^{N-1}\left(\frac{R}{r}\right) dx = \omega_N N(N-1)^{N-1}\,.$$

This means that $\{f_\varepsilon\}$ weakly converges in a weighted L^1 space to a kind of weighted Dirac mass. Actually the functions (18) seem to be solutions in $\Omega^\sharp$ to an equation which, in contrast with (19), is not homogeneous, but has a datum which can be seen as a weighted measure concentrated in the origin.

We conclude this section with the statement of the pointwise comparison result when

$$q_l < q < p\,. \tag{21}$$

We leave out the case $q = p$ owing to its peculiarity. The interested reader can refer to [19], [11], [26], [23], [24], [20], [21] for example.

Also in this case the feature is about the summability of f. It belongs to a Lorentz space strictly included in L^1.

We recall that a measurable function g, defined in Ω, belongs to the Lorentz space $L^{t,1}$ with $t > 1$ if

$$\|g\|_{t,1} = \int_0^{|\Omega|} g^*(s)\, s^{\frac{1}{t}-1}\, ds < +\infty\,.$$

We state the following result whose proof is in [27].

Theorem 3 *Let us suppose that* (2)*,* (5) *and* (21) *hold true. If* $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *is a weak solution to problem* (1) *with* f *bounded and*

$$\|f\|_{\frac{N(q-p+1)}{q},1} < K_3 \tag{22}$$

where

$$K_3 = (N\omega_N^{1/N})^{\frac{q}{q-p+1}} \left[\frac{1}{h}\left(1 - \frac{q}{N(q-p+1)}\right)\right]^{\frac{p-1}{q-p+1}}\,,$$

then

$$u^*(s) \leq \frac{K_3^{\frac{1}{p-1}}}{(N\omega_N^{1/N})^{\frac{p}{p-1}}} \frac{N(q-p+1)}{p-q} \left(s^{-\frac{p-q}{N(q-p+1)}} - |\Omega|^{-\frac{p-q}{N(q-p+1)}}\right)\,.$$

3 A priori estimates and existence results

In this section we state a priori estimates and existence results for solutions to problem (1) under the sharp smallness assumptions on f stated in the previous section.

We begin by considering case (7).

We can state the following result where a priori estimates of some Marcinkiewicz norms of the solution u and of its gradient are given (see also [22] for a different approach).

We recall that a function g is weakly L^t, with $t > 1$, or, equivalently, belongs to the Marcinkiewicz space L^t_w if

$$\|g\|_{L^t_w} = \sup_s \left(g^*(s)\, s^{-\frac{1}{t}} \right) < +\infty\,.$$

Theorem 4 *Let us suppose that* (2), (5) *and* (7) *hold true. If* $u \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ *is a weak solution to problem* (1) *with* f *bounded and if* (10) *is satisfied then*

$$\|u\|_{L_w^{\frac{N(p-1)}{N-p}}} \leq C_1(p, q, N, |\Omega|, h, \|f\|_{L^1}) \tag{23}$$

$$\|\,|\nabla u|\,\|_{L_w^{\frac{N(p-1)}{N-1}}} \leq C_2(p, q, N, |\Omega|, h, \|f\|_{L^1})\,. \tag{24}$$

Sketch of the proof. The comparison result (11) immediately implies

$$u^*(s) \leq \frac{1}{(N\omega_N^{1/N})^{\frac{p}{p-1}}} \frac{N(p-1)}{N-p} \frac{\|f\|_{L^1}^{\frac{1}{p-1}}}{\left[1 - \left(\dfrac{\|f\|_{L^1}}{K_1}\right)^{\frac{q-p+1}{p-1}}\right]^{\frac{1}{q-p+1}}} s^{-\frac{N-p}{N(p-1)}}$$

from which we get (23).

By choosing suitable test functions, built on the level sets of u, and adapting the techniques used in [28] (see also [29]), we can prove

$$|\nabla u|^*(s) \leq C(N, p, q, |\Omega|, \|f\|_{L^1}) s^{-\frac{N-1}{N(p-1)}}$$

and then (24). □

Here we just give the statement of the existence result. For the proof you can refer to [22].

Theorem 5 *Under the hypotheses* (2)-(5) *and* (7), *if* $f \in L^1(\Omega)$ *and condition* (10) *holds, then there exists at least a SOLA* u *to problem* (1) *which satisfies* (23) *and* (24).

Now, let us consider case (21).

As above we can obtain sharp a priori estimates and then existence results. Here we just give the statements of the results. We remark that we must distinguish the following two subcases

(i) $q_l < q < p - 1 + \dfrac{p}{N}$

(ii) $p - 1 + \dfrac{p}{N} \leq q < p\,.$

Theorem 6 *Let us suppose that* (2)-(5) *and* (i) *hold true. Let* $u \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ *be a weak solution to the problem* (1) *with* f *bounded. If* (22) *is satisfied then*

$$\|u\|_{L_w^{\frac{N(q-p+1)}{p-q}}} \leq C_3 \tag{25}$$

and

$$\|\,|\nabla u|\,\|_{L^t} \leq C_4 \tag{26}$$

with $t < N(q-p+1)$, *where* C_3, C_4 *are positive constants depending on* p, q, N, $|\Omega|$, h *and the norm of* f *in the Lorentz space* $L^{\frac{N(q-p+1)}{q},1}$.

Theorem 7 *Let us suppose that* (2)-(5) *and* (ii) *hold true. Let* $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be a weak solution to the problem* (1) *with* f *bounded. If* (22) *is satisfied then*

$$\| |\nabla u| \|_{L^p} \leq C_5 \tag{27}$$

and

$$\|u\|_{L^t} \leq C_6 \tag{28}$$

with t *less than the Sobolev conjugate of* $N(q-p+1)$*. The positive constants* C_5, C_6 *depend on* p*,* q*,* N*,* $|\Omega|$*,* h *and the norm of* f *in the Lorenz space* $L^{\frac{N(q-p+1)}{q},1}$.

Now we give the statements of the existence results which can be deduced by the previous a priori estimates. The proof is standard (see [22]).

Theorem 8 *Let us assume* (2)-(5) *and* (i) [(ii) *respectively*]*. If* f *belongs to* $L^{\frac{N(q-p+1)}{q},1}$ *and* (22) *is satisfied, then there exists at least a SOLA to the problem* (1)*. Moreover* (25), (26) [(28), (27) *respectively*] *hold.*

A few words about the limit case $q = q_l$ are in order.

If f satisfies (16) we can prove that u^* has the following behaviour near zero

$$u^*(s) = 0\left(s^{1-\frac{p(N-1)}{N(p-1)}} |\log(s)|^{-\frac{N-1}{p-1}}\right)$$

while the rearrangement of the gradient of u behaves in zero in such a way that

$$\int_0 (|\nabla u|^*(s))^q |\log(s)|^\tau \, ds < +\infty$$

with $\tau < N - 1$.

As in the previous cases these facts are the main ingredients for proving the existence of a SOLA u to the problem (1).

Remark 1 Uniqueness results for problem (1) are proved in [30], [31] and [32] when

$$q \leq p - 1 + \frac{p}{N}. \tag{29}$$

In [32] the uniqueness is proved under the sharp assumptions on the size of the norms of f described in section 2.

On the contrary, uniqueness is unknown when q exceeds the value on the right hand side in (29).

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