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Some Problems on Boundary Controllability for PDE's†

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a Mario con profondo affetto

Summary

This paper deals with null boundary controllability of two PDE's in one-dimensional x-space and time $t > 0$, modeling a composite solid with different physical properties in each semi-infinite layer.
Interface conditions are assumed.
Furthermore, it is pointed out possible extensions of such an investigation to two-dimensional (x, y)-space.

Key words: *null boundary control, heat equation, Schrödinger equation, interface condition, Gevrey class.*

Riassunto

Questo lavoro tratta problemi di controllabilità alla frontiera per equazioni alle derivate parziali. Si studia in modo specifico il caso di una funzione $u = u(x, t)$, $x \in \mathbb{R}$ e $t > 0$, soddisfacente un'equazione variazionale in $\mathbb{R} \times (0, +\infty)$, che modella un solido composito con differenti proprietà fisiche in differenti strati. Sono assunte condizioni di interfaccia. Inoltre, sono presentate estensioni di tale indagine al caso di funzioni di due variabili spaziali $u = u(x, y, t)$.

Parole chiave: *Controllo alla frontiera, equazione del calore, equazione di Schrödinger, condizioni di interfaccia, funzioni di Gevrey.*

1 Introduction

Let $a,\ b \in \mathbb{C} \setminus \{0\}$, $\mathrm{Re}\, a \geq 0$, and $\mathrm{Re}\, b \geq 0$. Let $\alpha \in \mathbb{R}$, $k_1 \neq k_2 \in (\mathbb{R} \setminus \{0\})$. Assume $\frac{\alpha k_1}{\sqrt{a}} + \frac{k_2}{\sqrt{b}} \neq 0$. The problem we will deal with is the following:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(c(x)\frac{\partial u}{\partial x}\right), \qquad x \in \mathbb{R}\ ,\ t > 0\,, \tag{1}$$

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where $c(x) = \begin{cases} a\,, & \text{for } x < 0 \\ b\,, & \text{for } x > 0 \end{cases}$

and the function $u = u(x,t)$ subjected to the initial condition:

$$u(x,0) = f(x)\,,\ \ x \neq 0 \tag{2}$$

with $f \in C^0(\mathbb{R} \setminus \{0\})$, $\quad f \equiv 0 \quad$ if $|x| \geq R > 0$.

The following interface conditions are requested:

$$u(0^-,t) = \alpha u(0^+,t)\,, \qquad \text{for any } t \geq 0 \tag{3}$$

$$k_1 u_x(0^-,t) = k_2 u_x(0^+,t)\,, \quad \text{for any } t \geq 0. \tag{4}$$

Moreover:

$$u(x_1,t) = h_1(t)\,,\ x_1 < 0 \quad \text{and} \quad u(x_2,t) = h_2(t)\,,\ x_2 > 0. \tag{5}$$

For the above described problem (1)-(5), we are interested in the null boundary controllability of the system; that is, given $T > 0$, we look for proper controls $h_1(t)$ and $h_2(t)$ such that, given initial data f in the appropriate space, the solution $u(x,t)$ of the system vanishes for $t \geq T$.
As a matter of fact, since the 70's a large number of authors has studied boundary controllability problems and applied different methods.
In this context, it is worthy to recall results of meaningful interest as those, among the others, by Avalos and Lasiecka [3] on null-controllability and by Lasiecka and Triggiani [5], [6], where the method of multipliers was employed to obtain boundary controllability results for the Schrödinger equation. Similar null controllability problems for one dimensional models where also analyzed in [11] and [12].

In 1985, W. Littman [7] presented "direct methods" to solve boundary controllability problems. In this framework, we refer to papers by Littman and Taylor [8], [9], [10].
For our study we will adopt the approach of direct methods, introduced by Littman [7], consisting of four steps. The first step is performed by solving the pure initial value problem (1)-(2) under the interface conditions (3)-(4). This aim, actually, will be accomplished by the theorem 2.1 in section 2. Then, in section 3, the second step will be described: precisely, the solution $u = u(x,t)$ will be multiplied by a cut-off function, belonging to a suitable Gevrey class. Section 4 will deal with the third step solving related side-ways Cauchy problems. Finally, the fourth step: in section 5, the control functions h_i will be found.

2 The Pure Initial Value Problem (1)-(4)

First of all, we look for solutions to the problem (1)-(4). To this end, we assume that $u = u(x,t)$ satisfies:

$$(*) \quad \begin{cases} |u(x,t)| \leq k e^{ht}\,, \quad k > 0\,, h \geq 0\,, \quad \text{for } x \neq 0\,, t > 0 \\ \\ \lim_{|x|\to\infty}\ (max_{[0,T]}\,|u(x,t)|) = 0\,, \quad \text{for every} \quad T > 0\,. \end{cases}$$

The following Theorem holds.

Theorem 2.1 *The problem (1)-(4), $(*)$ being valid, has a unique solution $u \in C^\infty((\mathbb{R} \setminus \{0\}) \times (0,+\infty))$, given by:*

$$\begin{aligned} u(x,t) = \quad & \alpha \frac{\frac{k_2}{b} \int_0^{+\infty} \frac{1}{\sqrt{\pi t}}\, exp\left[-\frac{1}{4t}\left(\frac{\xi}{\sqrt{b}} - \frac{x}{\sqrt{a}}\right)^2\right] f(\xi) d\xi + \frac{k_1}{a} \int_{-\infty}^{0} \frac{1}{\sqrt{\pi t}}\, exp\left[-\frac{1}{4t}\left(\frac{\xi + x}{\sqrt{a}}\right)^2\right] f(\xi) d\xi}{\alpha \frac{k_1}{\sqrt{a}} + \frac{k_2}{\sqrt{b}}} + \\ & + \frac{1}{a} \int_0^x d\xi \int_\xi^{-\infty} \frac{\xi - \xi_1}{2\sqrt{\pi a t^3}}\, exp\left[-\frac{1}{4at}(\xi - \xi_1)^2\right] f(\xi_1) d\xi_1\,, \quad in\ (-\infty,0) \times (0,+\infty); \end{aligned} \tag{6}$$

and

$$u(x,t) = \frac{\frac{k_2}{b}\int_0^{+\infty} \frac{1}{\sqrt{\pi t}}\, exp\left[-\frac{1}{4t}\left(\frac{x+\xi}{\sqrt{b}}\right)^2\right] f(\xi)d\xi + \frac{k_1}{a}\int_{-\infty}^0 \frac{1}{\sqrt{\pi t}}\, exp\left[-\frac{1}{4t}\left(\frac{x}{\sqrt{b}}-\frac{\xi}{\sqrt{a}}\right)^2\right] f(\xi)d\xi}{\alpha\frac{k_1}{\sqrt{a}}+\frac{k_2}{\sqrt{b}}} +$$
$$+\frac{1}{b}\int_0^x d\xi \int_\xi^{+\infty} \frac{\xi_2-\xi}{2\sqrt{\pi a t^3}}\, exp\left[-\frac{1}{4bt}(\xi_2-\xi)^2\right] f(\xi_2)d\xi_2\,, \quad in\ (0,+\infty)\times(0,+\infty). \tag{7}$$

Proof. Assume that a solution u in the above class does exist. Conditions $(*)$ guarantee that the Laplace transform

$$U(x,s) = \int_0^{+\infty} e^{-st}u(x,t)dt = \mathcal{L}[u(x,\cdot),s]$$

is defined in $(\mathbb{R}\setminus\{0\})\times\{s\in\mathbb{C}\,,\ Re\, s > h\}$ and it is holomorphic in s.
By using Laplace transform, the problem (1)-(4) becomes:

$$sU(x,s) - f(x) = c(x)U_{xx}(x,s)\,, \qquad x\neq 0\,,\quad Re\, s > h \tag{8}$$
$$U(0^-,s) = \alpha U(0^+,s)\,, \qquad Re\, s > h \tag{9}$$
$$k_1 U_x(0^-,s) = k_2 U_x(0^+,s)\,, \qquad Re\, s > h \tag{10}$$
$$\lim_{x\to\pm\infty} U(x,s) = 0\,, \qquad Re\, s > h\,. \tag{11}$$

One may easily prove that problem (8)-(11) has a unique solution. Indeed, if U_1 and U_2 were two solutions of problem (8)-(11), then their difference V would satisfy (8)-(11) with $f \equiv 0$. Solving the ordinary differential equation (8) and taking into account (9), (10), (11), one easily infers $V \equiv 0$.
As far as the existence of solution is concerned, let us look for a solution of (8)-(11) of the form:
for $x < 0$,

$$U(x,s) = c_1^-\, e^{\sqrt{\frac{s}{a}}x} + \frac{1}{2\sqrt{as}}\int_0^x g_1(\xi,s)\, e^{\sqrt{\frac{s}{a}}(x-\xi)}d\xi\,, \tag{12}$$

for $x > 0$,

$$U(x,s) = c_2^+\, e^{-\sqrt{\frac{s}{b}}x} + \frac{1}{2\sqrt{bs}}\int_0^x g_2(\xi,s)\, e^{-\sqrt{\frac{s}{b}}(x-\xi)}d\xi\,, \tag{13}$$

with c_1^- and c_2^+ constants to be determined.
To satisfy equation (8) one is led to solve:

$$\begin{cases} -\frac{1}{a}f(x) = \frac{1}{2\sqrt{as}}\left(g_1'(x,s) - \sqrt{\frac{s}{a}}g_1(x,s)\right) \quad , & \text{if } x<0 \\ -\frac{1}{b}f(x) = \frac{1}{2\sqrt{bs}}\left(g_2'(x,s) - \sqrt{\frac{s}{b}}g_2(x,s)\right) \quad , & \text{if } x>0. \end{cases} \tag{14}$$

From (14) it turns out that:

$$\begin{cases} g_1(x,s) = 2\int_x^{-\infty} \sqrt{\frac{s}{a}}\, e^{-\sqrt{\frac{s}{a}}(x-\xi)} f(\xi)d\xi \quad , & \text{if } x<0 \\ g_2(x,s) = 2\int_x^{+\infty} \sqrt{\frac{s}{b}}\, e^{-\sqrt{\frac{s}{b}}(x-\xi)} f(\xi)d\xi \quad , & \text{if } x>0. \end{cases} \tag{15}$$

Notice that both g_1 and g_2 have compact support.
On the other hand, by the interface conditions (9) and (10), one gets:

$$\begin{cases} c_1^- = \alpha c_2^+ \\ \left(\alpha k_1\sqrt{\frac{s}{a}} + k_2\sqrt{\frac{s}{b}}\right)c_2^+ = \frac{k_2}{2\sqrt{bs}}g_2(0,s) - \frac{k_1}{2\sqrt{as}}g_1(0,s) \quad . \end{cases} \tag{16}$$

Therefore, the solution of problem (8) - (11) is represented as follows:

$$\begin{aligned} U(x,s) = \ & \alpha \frac{\frac{k_2}{b}\int_0^{+\infty}\frac{1}{\sqrt{s}}\,exp\Big[-\sqrt{s}\Big(\frac{\xi}{\sqrt{b}}-\frac{x}{\sqrt{a}}\Big)\Big]f(\xi)d\xi+\frac{k_1}{a}\int_{-\infty}^{0}\frac{1}{\sqrt{s}}\,exp\Big[\sqrt{s}\Big(\frac{\xi+x}{\sqrt{a}}\Big)\Big]f(\xi)d\xi}{\alpha\frac{k_1}{\sqrt{a}}+\frac{k_2}{\sqrt{b}}}+ \\ & +\frac{1}{a}\int_0^x d\xi\int_\xi^{-\infty} exp\left[-\sqrt{s}\frac{\xi-\xi_1}{\sqrt{a}}\right]f(\xi_1)d\xi_1\,, \qquad \text{if } x<0 \end{aligned} \tag{17}$$

and

$$\begin{aligned} U(x,s) = \ & \frac{\frac{k_2}{b}\int_0^{+\infty}\frac{1}{\sqrt{s}}\,exp\Big[-\sqrt{s}\Big(\frac{\xi+x}{\sqrt{b}}\Big)\Big]f(\xi)d\xi+\frac{k_1}{a}\int_{-\infty}^{0}\frac{1}{\sqrt{s}}\,exp\Big[-\sqrt{s}\Big(\frac{x}{\sqrt{b}}-\frac{-xi}{\sqrt{a}}\Big)\Big]f(\xi)d\xi}{\alpha\frac{k_1}{\sqrt{a}}+\frac{k_2}{\sqrt{b}}}+ \\ & +\frac{1}{b}\int_0^x d\xi\int_\xi^{-\infty} exp\left[-\sqrt{s}\frac{\xi_2-\xi}{\sqrt{b}}\right]f(\xi_2)d\xi_2\,, \qquad \text{if } x>0\,. \end{aligned} \tag{18}$$

By inverse Laplace transform, we get the function $u = u(x,t)$, given by (6) and (7), solution to the problem (1)-(4). The proof of the theorem is so completed. □

To solve the null boundary controllability of the system, the following result is valid [1].

Theorem 2.2 *Let $x_1, x_2 \in \mathbb{R},\ x_1 < 0,\ x_2 > 0$. Let $T > 0$. Then, given in $[x_1, x_2]$ initial data f belonging to the space of piecewise continuous functions, there exist boundary control functions $h_1(t)$ and $h_2(t)$, which, if applied at x_1 and x_2, steer the solution $u = u(x,t)$ of our problem to zero in the interval $[0,T]$.*

For the proof of this theorem, the reader is referred to section 5.

3 The 2^{nd} Step

Let $\psi = \psi(t)$ be a cut-off function:

$$\psi(t) = \begin{cases} 1 & , \quad \text{for} \quad t \le \frac{T}{2} \\ 0 & , \quad \text{for} \quad t \ge T \end{cases} ,$$

belonging to a Gevrey class γ^δ (see [4]); choose ψ so that $\psi \in \gamma^{\frac{3}{2}}$, as in [1], i.e.

$$|\psi^{(n)}(t)| \le \Gamma\left(\frac{3}{2}\,n\right) c\ \theta^n\ , \ \text{for all} \quad n = 1, 2, \ldots$$

with c, θ indipendent of n and Γ the Gamma function.
Now, multiply $u = u(x,t)$ by $\psi(t)$ and get:

$$\begin{array}{lll} \left(a\,\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right)(u^-\psi) = -u^-(x,t)\psi'(t) & \quad \text{for} & \quad x \le 0^-,\ t > 0 \\ \left(b\,\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right)(u^+\psi) = -u^+(x,t)\psi'(t) & \quad \text{for} & \quad x \ge 0^+,\ t > 0\,, \end{array}$$

where u^- and u^+ denote the solution u, respectively, for $x < 0$ and $x > 0$.

We remark that the function $u(x,t)\,\psi(t)$ has the advantage that it vanishes for $t \ge T$, but at a price. Indeed, it generates the "garbage terms" $g^- = -u^-(x,t)\,\psi'(t)$ and $g^+ = -u^+(x,t)\psi'(t)$, respectively, on the left and the right of $x = 0$.

4 The 3^{rd} Step

Let us recall that the "garbage terms" $g^-(x,t)$ and $g^+(x,t)$ are analytic functions of x and belong to a Gevrey class as functions of t. Note, also, that g^- and g^+ vanish outside $\left[\frac{T}{2}, T\right]$.
To get rid of these terms, one has to solve the non-homogeneous equations:

$$\begin{cases} a\,\frac{\partial^2 w^-}{\partial x^2} - \frac{\partial w^-}{\partial t} = g^- & \text{for} \quad x \le 0^-, \quad t > 0 \\ b\,\frac{\partial^2 w^+}{\partial x^2} - \frac{\partial w^-}{\partial t} = g^+ & \text{for} \quad x \ge 0^+, \quad t > 0\,, \end{cases}$$

with zero initial conditions for $w^\mp$ and $w_x^\mp$ on $x = 0$.
The solutions w^- and w^+ vanish outside $\left[\frac{T}{2}, T\right]$.

We apply, separately, for $x < 0$ and $x > 0$, Hörmander's result [4] on Cauchy problems for differential operators with constant coefficients, having Cauchy data belonging to Gevrey class $\gamma^\delta(1 < \delta \le 2)$. To this end, see theorem 5.7.3 in Hörmander [4]; also, the example on page 150.

5 The Proof of Theorem 2.2

Proof. Let $x_1 \in \mathbb{R}$, $x_1 < 0$ and $x_2 \in \mathbb{R}$, $x_2 > 0$. Let $[x_1, x_2]$ be the physical region and $T > 0$.

By the results of previous sections 2, 3 and 4, we are in a position to find the control functions h_1 and h_2 we are looking for. To this end, keeping in mind the results of previous sections, let us define:

$$\widetilde{\omega}_-(x,t) = u^-(x,t)\,\psi(t) - w^-(x,t) \qquad \text{for} \quad x \le 0^-,\ t > 0$$

and

$$\widetilde{\omega}_+(x,t) = u^+(x,t)\,\psi(t) - w^+(x,t) \qquad \text{for} \quad x \ge 0^+,\ t > 0.$$

The functions $\widetilde{\omega}_-$ and $\widetilde{\omega}_+$ satisfy (1), (2) and vanish for $t \ge T$. The boundary controls are then given by:

$$h_1(t) = \widetilde{\omega}_-(x_1, t)\,, \quad x_1 < 0\,,$$

and

$$h_2(t) = \widetilde{\omega}_+(x_2, t)\,, \quad x_2 > 0\,.$$

□

6 Final Remarks

We point out that the investigation, performed in this paper in one x-space dimension, may be carried out to a proper study in two (x, y)-space dimensions.

(i) One may study, for instance, an equation like

$$u_t = c(x,y)(u_{xx} + u_{yy}) \qquad \text{in} \quad \mathbb{R}^2 \times (0, +\infty),$$

where $c(x,y) = a$ in the half-plane $x < 0$, for any $y \in \mathbb{R}$, and $c(x,y) = b$ in the half-plane $x > 0$, for any $y \in \mathbb{R}$, with a and b real or complex numbers, not zero, different each other, in general.
For instance, if $a = 1$ and $b = i$, the equation may be modeling phenomena in a composite solid composed by two parts governed, at the same time t, one of them by heat equation and the other by Schrödinger equation. The function $u = u(x,y,t)$ is required to satisfy initial condition $u(x,y,0) = f(x,y)$, with f a function with compact support in $\mathbb{R}^2$. Of course, interface conditions must be imposed for u and u_x.

(ii) One may study a problem describing different physical phenomena in a composite solid composed by four parts governed, at the same time t, by different equations. The coefficient $c(x, y)$ might assume different values, real or complex A_i in the quadrant Q_i of the plane $\mathbb{R}^2$, for $i = 1, 2, 3, 4$. Of course, initial conditions and suitable interface conditions should be required.

A paper of mine about problems in (i) (O. Arena, *A Problem of Boundary Controllability for a Plate*) will appear soon in a special volume, in honor of W. Littman, published by EECT (Evolution Equations and Control Theory). Studies on problems in the more general frame, referred in (ii), are in progress and will appear later.

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