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Degenerate Parabolic Problems in Turbulence Modelling†

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Dedicated to Prof. M. Marino on the occasion of his 70^{th} *birthday.*

Summary

In this paper, we consider one-equation models of turbulence with turbulent-viscosity $\nu_T = \ell\sqrt{k}$ (ℓ = length scale, k = mean turbulent kinetic energy). The following system of two parabolic equations represents a simplified model for the turbulent flow of an incompressible fluid through a pipe with cross-section $\Omega \subset \mathbb{R}^2$:

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(\sqrt{k}\nabla u\right) = 0, \quad \frac{\partial k}{\partial t} - \operatorname{div}\left((\mu + \sqrt{k})\nabla k\right) = \sqrt{k}|\nabla u|^2 - k\sqrt{k} \quad \text{in} \quad \Omega\times]0,T[,$$

where μ = const > 0. Here, the differential equation on the left is degenerate due to the coefficient $\sqrt{k}$.

We prove the existence of a weak solution (u, k) of this system under homogeneous boundary conditions and initial conditions $u(0) = u_0$ and $k(0) = k_0$. The pair (u, k) exhibits the phenomenon of turbulence as follows. If

$$\int_0^T \int_\Omega |\nabla u|^2 dxdt > 0,$$

then there exists a set $Q^* \subset \Omega\times]0,T[$ such that

$$\text{mes}\, Q^* > 0, \quad k > 0 \quad \text{a.e. in} \quad Q^*.$$

Key words: *Degenerate parabolic equations (35K65), weak solutions (35D30), turbulent-vicosity model (76F99), local energy equality (35D99)*

Riassunto

In questo articolo consideriamo modelli di turbolenza ad un'equazione con viscosità di turbolenza $\nu_T = \ell\sqrt{k}$ (ℓ = scala di lunghezza, k = energia cinetica media di turbolenza). Il seguente sistema di due equazioni paraboliche rappresenta un modello semplificato per il flusso turbolento di un fluido incomprimibile attraverso un tubo con sezione trasversale $\Omega \subset \mathbb{R}^2$:

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(\sqrt{k}\nabla u\right) = 0, \quad \frac{\partial k}{\partial t} - \operatorname{div}\left((\mu + \sqrt{k})\nabla k\right) = \sqrt{k}|\nabla u|^2 - k\sqrt{k} \quad \text{in} \quad \Omega\times]0,T[,$$

dove μ = cost > 0. Qui l'equazione differenziale a sinistra è degenere a causa del coefficiente $\sqrt{k}$.

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Noi proviamo l'esistenza di una soluzione debole (u, k) di questo sistema con condizioni al bordo omogenee e condizioni iniziali $u(0) = u_0$ e $k(0) = k_0$. Tale soluzione (u, k) mostra il fenomeno di turbolenza nel seguente modo. Se

$$\int_0^T \int_\Omega |\nabla u|^2 dxdt > 0,$$

allora esiste un insieme $Q^* \subset \Omega\times]0, T[$ tale che

$$\text{mis}\, Q^* > 0, \quad k > 0 \quad \text{q. o. in} \quad Q^*.$$

Parole chiave: *Equazioni paraboliche degeneri (35K65), soluzioni deboli (35D30), modello di viscosità di turbolenza (76F99), uguaglianza di energia locale (35D99)*

Contents

1. Introduction

1.1 Turbulent-viscosity models

Let $\Omega \subset \mathbb{R}^3$ denote a domain, let $0 < T < +\infty$ and set $Q_T := \Omega\times]0, T[$. We consider the following system of PDEs

$$\text{div}\, \boldsymbol{u} = 0 \quad \text{in} \quad Q_T, \tag{1.1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \text{div}\Big((\nu + \ell\sqrt{k})\boldsymbol{D}(\boldsymbol{u})\Big) - \nabla p + \boldsymbol{f} \quad \text{in} \quad Q_T, \tag{1.2}$$

$$\frac{\partial k}{\partial t} + \boldsymbol{u} \cdot \nabla k = \text{div}\Big((\mu + \ell\sqrt{k})\nabla k\Big) + \ell\sqrt{k}|\boldsymbol{D}(\boldsymbol{u})|^2 - \frac{k\sqrt{k}}{\ell} \quad \text{in} \quad Q_T, \tag{1.3}$$

where the unknowns are

$\boldsymbol{u} = (u_1, u_2, u_3)$ mean velocity field,

p = modified mean pressure,

k = mean turbulent kinetic energy.

If $(\boldsymbol{u}, p, k)$ is a solution to (1.1)–(1.3) then the turbulent motion of the fluid is characterized by the velocity field $\boldsymbol{U} = \boldsymbol{u} + \widetilde{\boldsymbol{u}}$, where $\widetilde{\boldsymbol{u}}$ denotes the fluctuation of the motion. The mean turbulent kinetic energy is then specified by

$$k = \frac{1}{2}\overline{|\widetilde{\boldsymbol{u}}|^2} \quad \Big(= \text{ mean of } \frac{1}{2}|\widetilde{\boldsymbol{u}}|^2 \Big).$$

Further notations in (1.2) and (1.3) are

$$\boldsymbol{D}(\boldsymbol{u}) = \{D_{ij}(\boldsymbol{u})\}_{i,j=1,2,3} = (\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top), \quad |\boldsymbol{D}(\boldsymbol{u})|^2 = D_{ij}(\boldsymbol{u})D_{ij}(\boldsymbol{u})\,^{1}$$
$$\boldsymbol{f} = \text{ given external force}$$
$$\nu = \text{const} \geq 0, \quad \mu = \text{const} \geq 0.$$

The turbulent-viscosity ν_T of the fluid is modelled by the Boussinesq hypothesis

$$\nu_T = \ell\sqrt{k}, \quad \ell = \text{turbulent length scale (mixing length)}.$$

In (1.3), the term $\ell\sqrt{k}|\boldsymbol{D}(\boldsymbol{u})|^2$ represents the rate of transferring turbulent kinetic energy from the mean flow to the turbulence, while the sink term $-\frac{k\sqrt{k}}{\ell}$ models the decay of energy (dissipation) of the turbulence. The characteristic length scale ℓ depends on the flow under consideration and its Reynolds number Re. Thus, ℓ is an unspecified part of (1.1)–(1.3).
For a detailed discussion of mean-flow equations and turbulent-viscosity models we refer to [16], [17], [23]. ■

In case of the fully developed turbulence, the coefficients $\nu = \mu = \frac{1}{\text{Re}}$ can be neglected. Then (1.2), (1.3) take the form

$$\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \operatorname{div}\Big(\ell\sqrt{k}\boldsymbol{D}(\boldsymbol{u})\Big) - \nabla p + \boldsymbol{f} \quad \text{in} \quad Q_T, \tag{1.4}$$

$$\frac{\partial k}{\partial t} + \boldsymbol{u}\cdot\nabla k = \operatorname{div}\Big(\ell\sqrt{k}\nabla k\Big) + \ell\sqrt{k}|\boldsymbol{D}(\boldsymbol{u})|^2 - \frac{k\sqrt{k}}{\ell} \quad \text{in} \quad Q_T, \tag{1.5}$$

in Q_T, respectively. System (1.1), (1.4), (1.5) is usually called *Prandtl's (1945) one-equation model of turbulence.* We notice that equ. (1.5) has been directly postulated by Prandtl [20] for one space dimension (cf. also [1], [18], [19] for more details). Moreover, (1.4) does not explicitly occur in [20]. Indeed, Prandtl only mentioned that the rate of strain $\boldsymbol{\tau} = \ell\sqrt{k}\boldsymbol{D}(\boldsymbol{u})$ can be "determined by the velocity field $\boldsymbol{u}$ from the Euler equation to which the term $\operatorname{div}(\ell\sqrt{k}\boldsymbol{D}(\boldsymbol{u}))$ is added" (cf. [20; p. 11]); notice that $\boldsymbol{\tau}_{ij}D_{ij}(\boldsymbol{u}) = \ell\sqrt{k}|\boldsymbol{D}(\boldsymbol{u})|^2$). ■

Remark 1.1 Let Ω be bounded. The following assumptions on the turbulent length scale ℓ include many examples which are widely used in the literature:

$$\ell \in C(\overline{\Omega}); \quad \ell(x) > 0 \quad \forall\, x \in \Omega, \quad \ell(x) = 0 \quad \forall\, x \in \partial\Omega. \tag{1.6}$$

The function

$$\ell(x) = \kappa_0(\operatorname{dist}(x, \partial\Omega))^\alpha, \quad x \in \overline{\Omega} \quad (\kappa_0 = \text{const} > 0, \alpha = \text{const} > 0)$$

clearly obeys (1.6) (cf., e. g., [16; pp. 302-306, 378-389 for $\alpha = 1$]).

Remark 1.2 A partial differential equation that governs the turbulent length scale ℓ, has been derived in [21].

[1] Throughout the paper a repeated index implies summation over 1, 2, 3.

1.2 Turbulent motion through a pipe

Let $\Omega_0 \subset \mathbb{R}^2$ be a bounded domain, and define $\Omega := \Omega_0 \times]0, a[$ (= "pipe with cross-section Ω_0 and length $0 < a < +\infty$"). Put

$$x = (x', x_3), \quad x' = (x_1, x_2) \in \Omega_0, \quad x_3 \in]0, a[.$$

Let $\boldsymbol{f} = \boldsymbol{0}$. In $Q_T = \Omega \times]0, T[$ we consider a flow that is driven by a pressure difference between $\Omega_0 \times \{0\}$ (= inlet) and $\Omega_0 \times \{a\}$ (= outlet), i. e., more specifically

$$p(x, t) = -g(t)x_3, \quad g(t) > 0 \quad \text{given on} \quad]0, T[.$$

This leads to unknown functions $(\boldsymbol{u}, k)$ of the structure

$$\boldsymbol{u}(x, t) = (0, 0, u_3(x', t)), \quad k(x, t) = k(x', t) \quad ((x, t) \in Q_T).$$

It follows

$$\operatorname{div} \boldsymbol{u} = 0, \quad (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{u} \cdot \nabla k = 0,$$

$$\boldsymbol{D}(\boldsymbol{u}) = \begin{pmatrix} 0 & 0 & \frac{1}{2}\partial_{x_1} u_3 \\ 0 & 0 & \frac{1}{2}\partial_{x_2} u_3 \\ \frac{1}{2}\partial_{x_1} u_3 & \frac{1}{2}\partial_{x_2} u_3 & 0 \end{pmatrix}, \quad |\boldsymbol{D}(\boldsymbol{u})|^2 = \frac{1}{2}|\nabla u_3|^2, \quad \nabla p = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}.$$

In addition, we assume that the turbulent length scale ℓ depends on $x' \in \Omega_0$ only, i. e.

$$\ell \in C(\overline{\Omega}_0); \quad \ell(x') > 0 \quad \forall\, x' \in \Omega_0, \quad \ell(x') = 0 \quad \forall\, x' \in \partial\Omega_0. \tag{1.7}$$

Thus, with the above assumptions on $\boldsymbol{u}$ and k, system (1.2), (1.3) takes the form

$$\frac{\partial u_3}{\partial t} = \frac{1}{2}\operatorname{div}\left((\nu + \ell\sqrt{k})\nabla u_3\right) + g \quad \text{in} \quad \Omega_0 \times]0, T[, \tag{1.8}$$

$$\frac{\partial k}{\partial t} = \operatorname{div}\left((\mu + \ell\sqrt{k})\nabla k\right) + \frac{1}{2}\ell\sqrt{k}|\nabla u_3|^2 - \frac{k\sqrt{k}}{\ell} \quad \text{in} \quad \Omega_0 \times]0, T[, \tag{1.9}$$

respectively. Analogously, system (1.4), (1.5) reads

$$\frac{\partial u_3}{\partial t} = \frac{1}{2}\operatorname{div}\left(\ell\sqrt{k}\nabla u_3\right) + g \quad \text{in} \quad \Omega_0 \times]0, T[, \tag{1.10}$$

$$\frac{\partial k}{\partial t} = \operatorname{div}\left(\ell\sqrt{k}\nabla k\right) + \frac{1}{2}\ell\sqrt{k}|\nabla u_3|^2 - \frac{k\sqrt{k}}{\ell} \quad \text{in} \quad \Omega_0 \times]0, T[, \tag{1.11}$$

respectively. ■

Remark 1.3 (*degenerate parabolic equations*) Since

$$\ell\sqrt{k} \geq 0 \quad \text{in} \quad \Omega_0 \in]0, T[, \quad \ell\sqrt{k} = 0 \quad \text{on} \quad \partial\Omega_0 \times [0, T],$$

the differential operator $\operatorname{div}(\ell\sqrt{k}\nabla(\cdot))$ on the right hand side of (1.10) and (1.11) is degenerated. The weak formulation of (1.10), (1.11) therefore has to make use of *Sobolev spaces with weight* ℓ. In addition, due to the physical meaning of the turbulence model under consideration, the weak solution (u_3, k) to (1.10), (1.11) must verify the condition

$$\int_0^T \int_{\Omega_0} \frac{k\sqrt{k}}{\ell} dxdt < +\infty.$$

1.3 A model problem

Throughout the remainder of the paper, let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We consider the following system of PDEs for the unknown scalar functions u and k

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(\sqrt{k}\nabla u\right) = g \quad \text{in} \quad Q_T, \tag{1.12}$$

$$\frac{\partial k}{\partial t} - \operatorname{div}\left((\mu + \sqrt{k})\nabla k\right) = \sqrt{k}|\nabla u|^2 - k\sqrt{k} \quad \text{in} \quad Q_T, \tag{1.13}$$

where g is a function defined on Q_T, and $\mu = \text{const} > 0$. in (1.12), the differential operator $\operatorname{div}(\sqrt{k}\nabla(\cdot))$ is *degenerated*. With regard to the nonlinear terms, (1.12), (1.13), system involves the same mathematical properties as system (1.8) (with $\nu = 0$), (1.9) does. Moreover, the proof of our main theorem (see Section 4) continues to hold for (1.8) ($\nu = 0$), (1.9) with turbulent length scales ℓ which satisfy (1.7) and, in addition,

$$\int_{\Omega_0} \frac{dx'}{\ell} < +\infty.$$

We complete (1.12), (1.13) by the boundary and initial conditions

$$u = 0, \quad \frac{\partial k}{\partial \boldsymbol{n}'} = 0 \quad \text{on} \quad \partial\Omega \times [0, T], \tag{1.14}$$

$$u = u_0, \quad k = k_0 \quad \text{on} \quad \Omega \times \{0\}, \tag{1.15}$$

where $\boldsymbol{n}'$ denotes the unit normal to $\partial\Omega$. The following figure illustrates the meaning of boundary conditions (1.14) with respect to the boundary of a pipe with cross-section Ω.

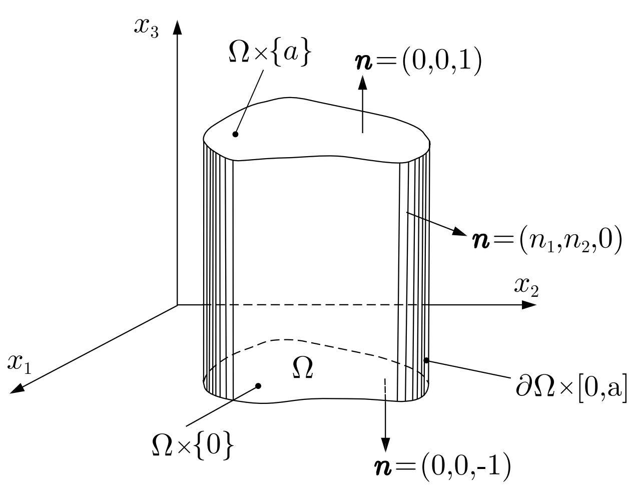

Figure 1: The "pipe $\Omega\times]0, a[$"

Let $\boldsymbol{n} = (n_1, n_2, n_3)$ denote the exterior unit normal to $\partial(\Omega\times]0, a[)$. For $\xi \in \mathbb{R}^3$, define $\xi_\tau := \xi - (\xi \cdot \boldsymbol{n})\boldsymbol{n}$. Let be $\boldsymbol{u}(x, t) = (0, 0, u_3(x', t))$ as above ($x = (x', x_3)$), where $x' \in \Omega$, $x_3 \in]0, a[$, and $t \in]0, T[$. Then, for every $t \in]0, T[$,

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{u}_\tau = \boldsymbol{u} \quad \text{on} \quad \partial\Omega \times [0, a],$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = \pm u_3, \quad \boldsymbol{u}_\tau = \boldsymbol{0} \quad \text{on} \quad \Omega \times \{0\} \text{ resp. } \Omega \times \{a\}.$$

Moreover,

$$(\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n})_\tau = 0 \quad \partial\Omega \times [0, a] \quad \Leftrightarrow \quad \frac{\partial u_3}{\partial \boldsymbol{n}'} = 0 \quad \text{on} \quad \partial\Omega,$$

where $\boldsymbol{n}' = (n_1, n_2)$. Thus, the boundary condition $\frac{\partial u_3}{\partial \boldsymbol{n}'} = 0$ on $\partial\Omega$ is equivalent to the Navier-slip condition on the vector field $\boldsymbol{u}(x, t) = (0, 0, u_3(x', t))$ on $\partial\Omega \times [0, a]$. The boundary condition $\frac{\partial k}{\partial \boldsymbol{n}'} = 0$ on $\partial\Omega$ means that there is no flux of k through $\partial\Omega$. ■

In [9; pp. 203-204], the author considers a system of PDEs for two scalar functions which is more complex than (1.12), (1.13), but does not include a degenerate parabolic equation like (1.12) in [4], [8] the authors establish the existence of weak solutions to a general class of turbulent-viscosity models in three dimensions of space with coefficients $\nu_T = \nu_0 + \nu(k)$ ($\nu_0 = \text{const} > 0$), where $0 \le \nu(k) \le c_0 k^\alpha$ for all $k \in [0, +\infty[$ ($\alpha > 0$ appropriate).

2. Weak solutions of the model problem

2.1 Weak formulation

Let X denote a real normed vector space with norm $|\cdot|$, let X^* be the dual of X and let $\langle x^*, x\rangle_{X^*,X}$ denote the dual pairing between $x^* \in X^*$ and $x \in X$. The symbol $C_w([0,T];X)$ stands for the vector space of all mappings $u : [0,T] \to X$ such that the function $t \mapsto \langle x^*, x\rangle_{X^*,X}$ is continuous on $[0,T]$ whenever $x^* \in X^*$. Next, by $L^p(0,T;X)$ $(1 \le p \le +\infty)$ we denote the vector space of all equivalence classes of measurable mappings $u : [0,T] \to X$ such that the function $t \mapsto |u(t)|_X$ is in $L^p(0,T)$ (see, e. g. [2; chap. III, § 3; chap. IV, § 3], [5]).

Let $W^{1,p}(\Omega)$ $(1 \le p \le +\infty)$ denote the usual Sobolev space. Define

$$W_0^{1,2}(\Omega) := \{u \in W^{1,p}(\Omega)\,;\, u = 0 \text{ a. e. on } \partial\Omega\},$$
$$W^{-1,p'}(\Omega) := \text{ dual of } W_0^{1,p}(\Omega) \quad \Big(1 < p < +\infty, p' = \frac{p}{p-1}\Big).$$

■

We introduce the notion of weak solution to (1.12)–(1.15). For the sake of simplicity of our presentation, in what follows we assume $g = 0$.

Definition *Let $u_0 \in L^2(\Omega)$ and $k_0 \in L^1(\Omega)$. The pair (u,k) is called weak solution to (1.12)–(1.15) if*

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W_0^{1,2}(\Omega)), \tag{2.1}$$

$$k \in L^\infty(0,T;L^1(\Omega)) \cap L^{3/2}(Q_T), \quad k \ge 0 \quad \text{a. e. in } Q_T, \tag{2.2}$$

$$\nabla k \in [L^p(Q_T)]^2 \left(1 < p < \frac{4}{3}\right),\ k^{1/2}\nabla k \in [L^q(Q_T)]^2 \ (1 < q \le p), \tag{2.3}$$

$$k^{1/4}\nabla u \in [L^2(Q_T)]^2 \tag{2.4}$$

and

$$\left.\begin{aligned} &-\int_{Q_T} u\frac{\partial v}{\partial t} + \int_{Q_T} k^{1/2}\nabla u\cdot\nabla v = \int_\Omega u_0(x)v(x,0)dx\ ^2 \\ &\forall\, v \in L^2(0,T;W_0^{1,4}(\Omega)) \text{ such that } \frac{\partial v}{\partial t} \in L^2(Q_T), \quad v(\cdot,T) = 0, \end{aligned}\right\} \tag{2.5}$$

$$\left.\begin{aligned} &-\int_{Q_T} k\frac{\partial\varphi}{\partial t} + \int_{Q_T}(\mu + k^{1/2})\nabla k\cdot\nabla\varphi = \int_\Omega k_0(x)\varphi(x,0)dx + \int_{Q_T}(k^{1/2}|\nabla u|^2 - k^{3/2})\varphi \\ &\forall\,\varphi \in C^1([0,T];W^{1,q'}(\Omega)) \quad \text{such that} \quad \varphi(\cdot,T) = 0, \\ &\Big(q \text{ as in (2.3)},\ q' = \frac{q}{q-1}\Big). \end{aligned}\right\} \tag{2.6}$$

[2] For notational simplicity, in what follows we write $\int_E f$ in place of $\int_E f\,dxdt$ $(E \subset \mathbb{R}^3)$.

We notice that the integrability of ∇k in (2.3) is well-known from the theory of parabolic equations with right hand side in L^1 $\Big($i. e., $1 \le p < \dfrac{d+2}{d+1}$, where d =dimension of space$\Big)$. The integrability of $k^{1/2}\nabla k$ in (2.3) follows from an a-priori estimate on the appropriate solution $(u_\varepsilon, k_\varepsilon)$ which will be deduced in Section 4.1. It is easy to see that (2.1)–(2.4) guarantee the integrability of the functions under the integral signs of the integral relations in (2.5) and (2.6) as well.

To motivate the integral relation in (2.5), we consider a sufficiently regular solution (u, k) ($k \ge 0$ in Q_T) of (1.12)–(1.15). Let v be a smooth function in $\overline{\Omega} \times [0, T]$ such that $v = 0$ on $\partial\Omega \times [0, T]$ and $v(x, T) = 0$ for all $x \in \Omega$. We multiply each term in (1.12) by v, integrate over Q_T and integrate by parts. This gives (2.5). Observing the boundary condition $\dfrac{\partial k}{\partial \boldsymbol{n}'} = 0$ on $\partial\Omega\times[0, T]$, we deduce (2.6) from (1.13) by an analogous reasoning.

2.2 Existence of t-derivatives

Let (u, k) be a weak solution to (1.12)–(1.15). We show that both u and k have a first order t-derivative in the sense of distributions of $]0, T[$ into $W^{-1,4/3}(\Omega)$ and $(W^{1,q'}(\Omega))^*$ (q as in (2.3)), respectively.

To this end, we introduce some notations. Let X and Y be real normed spaces such that

$$X \subseteq Y \quad \text{continuously.}$$

Let $v \in L^p(0, T; X)$ and $w \in L^q(0, T; Y)$ $(1 \le p, q \le +\infty)$ satisfy

$$\int_0^T v(t)\alpha(t)dt = -\int_0^T w(t)\alpha'(t)dt \quad \text{in} \quad Y, \quad \forall\, \alpha \in C_c^\infty(]0, T[).$$

Then w is called the *derivative* (*of order* 1) *of* v *in sense of distributions of* $]0, T[$ *into* Y and denoted by v' (see, e. g., [3; Appendice], [5]). The element v' is uniquely determined

The following result is well-known. *For every* $v \in L^p(0, T; X)$ *with distributional derivative* $v' \in L^q(0, T; Y)$ $(1 \le p, q < +\infty)$ *there exists* $\widetilde{v} \in C([0, T]; Y)$ *such that*

$$\left.\begin{aligned} &\widetilde{v}(t) = v(t) \quad \textit{for a. e. } t \in [0, T], \\ &\|\widetilde{v}\|_{C([0,T];Y)} \le c\Big(\|v\|_{L^p(X)} + \|v'\|_{L^q(Y)}\Big)^{3} \quad (c = \text{const } \textit{independent of } v). \end{aligned}\right\} \tag{2.7}$$

Next, let H be a real Hilbert space with scalar product $(\cdot, \cdot)_H$. We suppose that $X \subset H$ continuously. Identifying H with its dual H^*, it follows

$$X \subset H \cong H^* \subset X^* \text{ continuously,} \quad (h, \xi)_H = \langle h, \xi\rangle_{X^*,X} \quad \forall\, h \in H, \forall\, \xi \in X.$$

Let X be reflexive and let $1 \le p, q < +\infty$. Let $v \in L^p(0, T; X)$. Then the equivalence of 1° and 2° is readily seen.

1° $\exists\, w \in L^q(0, T; X^*)$ *such that*

$$\int_0^T v(t)\alpha'(t)dt = \int_0^T w(t)\alpha(t)dt \quad \textit{in} \quad X^*, \quad \forall\, \alpha \in C_c^1(]0, T[)$$

(i. e., v possesses the distributional derivative $v' = -w$);

2° $\exists\, w \in L^q(0, T; X^*)$ *such that*

$$\int_0^T (v(t), \xi)_H \alpha'(t)dt = \int_0^T \langle w(t), \xi\rangle_{X^*,X}\alpha(t)dt \quad \forall\, \xi \in X, \quad \forall\, \alpha \in C_c^1(]0, T[). \tag{2.8}$$

[3] In what follows, we briefly write $\|\cdot\|_{L^q(X)}$ in place of $\|\cdot\|_{L^p(0,T;X)}$.

Finally, for later use (Section 3) we notice the following elementary result. *Let* $1 < p < +\infty$. *For every* $v \in L^p(0,T;X)$ *with distributional derivative* $v' \in L^{p'}(0,T;X^*)$ *there exists* $\widetilde{u} \in C([0,T];H)$ *such that*

$$\widetilde{u}(t) = u(t) \text{ for a. e. } t \in [0,T], \quad \|\widetilde{u}\|^2_{C([0,T];H)} \le 2\|u\|_{L^p(X)}\|u'\|_{L^{p'}(X^*)}. \tag{2.9}$$

■

We are now in a position to prove the following

Proposition 1 *Let* (u,k) *be a weak solution of (1.12)–(1.15). Then there exist the distributional derivatives*

$$u' \in L^2(0,T;W^{-1,4/3}(\Omega)), \quad k' \in L^1(0,T;(W^{1,q'}(\Omega))^*) \tag{2.10}$$

and there holds

$$\left.\begin{aligned} &u \in C_w([0,T];L^2(\Omega)), \quad u(0) = u_0 \text{ in } L^2(\Omega),\\ &\langle u'(t),\xi\rangle_{W^{-1,4/3},W^{1,4}_0} + \int_\Omega k^{1/2}\nabla u(t)\cdot\nabla\xi dx = 0\\ &\quad \text{for a. e. } t \in [0,T], \quad \forall\, \xi \in W^{1,4}_0(\Omega), \end{aligned}\right\} \tag{2.11}$$

$$\left.\begin{aligned} &k \in C([0,T];(W^{1,q'}(\Omega))^*), \quad k(0) = k_0 \text{ in } (W^{1,q'}(\Omega))^*, ^4\\ &\langle k'(t),\eta\rangle_{(W^{1,q'})^*,W^{1,q'}} + \int_\Omega (\mu + k^{1/2}(t))\nabla k(t)\cdot\nabla\eta dx =\\ &= \int_\Omega \Big(k^{1/2}(t)|\nabla u(t)|^2 - k^{3/2}(t)\Big)\eta dx \text{ for a. e. } t \in [0,T], \quad \forall\, \eta \in W^{1,q'}(\Omega)\\ &\ (q \text{ as in (2.3) }). \end{aligned}\right\} \tag{2.12}$$

By the separability of the Sobolev space $W^{1,p}(\Omega)$, the sets of measure zero of those $t \in [0,T]$ for which the functional relations in (2.11) and (2.12) fail, do not depend on $\xi \in W^{1,4}_0(\Omega)$ and $\eta \in W^{1,q'}(\Omega)$, respectively.

Proof of Proposition 1 Observing (2.2) we obtain for all $\xi \in W^{1,4}_0(\Omega)$ and a. e. $t \in [0,T]$

$$\begin{aligned} &\left|\int_\Omega k^{1/2}(t)\nabla u(t)\cdot\nabla\xi dx\right| \le\\ &\le \|k\|^{1/4}_{L^\infty(L^1)}\Big(\int_\Omega k^{1/2}(t)|\nabla u(t)|^2 dx\Big)^{1/2}\Big(\int_\Omega |\nabla\xi|^4 dx\Big)^{1/4}. \end{aligned}$$

By (2.4), the function $t \mapsto \int_\Omega k^{1/2}(t)|\nabla u(t)|^2 dx$ is in $L^2(0,T)$. Hence, there exists $w \in L^2(0,T;W^{-1,4/3}(\Omega))$ such that

$$\int_\Omega k^{1/2}(t)\nabla u(t)\cdot\nabla\xi dx = \langle w(t),\xi\rangle_{W^{-1,4/3},W^{1,4}_0} \quad \text{for a. e.} \quad t \in [0,T]$$

(notice that the measurability of $w : [0,T] \to W^{-1,4/3}(\Omega)$ follows from Pettis' theorem).

[4] Here we have identified $k_0 \in L^1(\Omega)$ with the element in $(W^{1,q'}(\Omega))^*$ (again denoted by k_0) which is defined by $\langle k_0,\eta\rangle_{(W^{1,q'})^*,W^{1,q'}} = \int_\Omega k_0\eta dx \quad \forall\, \eta \in W^{1,q'}(\Omega)$.

Let $\alpha \in C^\infty([0,T])$, $\mathrm{supp}(\alpha) \subset]0,T[$. The function $v(x,t) = \xi(x)\alpha(t)$ $(x,t) \in \overline{\Omega}\times]0,T[)$ being admissible in (2.5), the integral relation takes the form

$$\int_0^T (u(t),\xi)_{L^2}\alpha'(t)dt = \int_0^T \langle w(t),\xi\rangle_{W^{-1,4/3},W_0^{1,4}}\alpha(t)dt,$$

i. e. (2.8) holds with $X = W_0^{1,4}(\Omega), H = L^2(\Omega)$. Hence, the distributional derivative $u'(= -w) \in L^2(0,T;W^{-1,4/3}(\Omega))$ exists (cf. (2.10)), and the functional relation in (2.11) holds for a. e. $t \in [0,T]$. This follows by a routine argument. Moreover, there exists a representative of u (not relabelled) that is in $C([0,T];W^{-1,4/3}(\Omega))$ (cf. (2.7)). Thus, by (2.1), $u \in C_w([0,T];L^2(\Omega))$.

We prove that $u(0) = u_0$ in $L^2(\Omega)$. To this end, let $\xi \in W_0^{1,4}(\Omega)$ and fix $\zeta \in C^1([0,1])$ such that $\zeta(T) = 0$ and $\zeta(0) = 1$. Then

$$\int_0^T \langle u'(t),\xi\rangle_{W^{-1,4/3},W_0^{1,4}}\zeta(t)dt + \int_0^T (u(t),\xi)_{L^2}\zeta'(t)dt = -(u(0),\xi)_{L^2}. \tag{2.13}$$

On the other hand, inserting the function $(x,t) \longmapsto \xi(x)\zeta(t)$ $((x,t) \in \overline{\Omega}\times[0,T])$ into (2.5) it follows

$$-\int_0^T (u(t),\xi)_{L^2}\zeta'(t)dt + \int_0^T \Big(\int_\Omega k^{1/2}(t)\nabla u(t)\cdot\nabla\xi dx\Big)dt = \int_\Omega u_0\xi dx. \tag{2.14}$$

Combining (2.11), (2.13) and (2.14) we find

$$\int_\Omega u_0\xi dx = \int_\Omega u(x,0)\xi dx, \quad \xi \in W_0^{1,4}(\Omega).$$

Whence the claim.

To establish the existence of $k' \in L^2(0,T;(W^{1,q'}(\Omega))^*)$ (q as in (2.3)), we notice that for every $\eta \in W^{1,q'}(\Omega)$, from (2.3) and (2.4) it follows

$$\begin{aligned}
&\Big|\int_\Omega \Big(\mu + k^{1/2}(t)\Big)\nabla k(t)\cdot\nabla\eta dx - \int_\Omega \Big(k^{1/2}(t)|\nabla u(t)|^2 - k^{3/2}(t)\Big)\eta dx\Big| \le \\
&\le \Big(\int_\Omega \Big((\mu + k^{1/2}(t))|\nabla k(t)|\Big)^q dx\Big)^{1/q}\Big(\int_\Omega |\nabla\eta|^{q'}dx\Big)^{1/q'} + \\
&\qquad + \int_\Omega \Big(k^{1/2}(t)|\nabla u(t)|^2 + k^{3/2}(t)\Big)dx \max_{\overline{\Omega}}|\eta|
\end{aligned}$$

for a. e. $t \in [0,T]$. Hence, there exists $z \in L^1(0,T;(W^{1,q'}(\Omega))^*)$ such that

$$\int_\Omega \Big(\mu + k^{1/2}(t)\Big)\nabla k(t)\cdot\nabla\eta dx - \int_\Omega \Big(k^{1/2}(t)|\nabla u(t)|^2 - k^{3/2}(t)\Big)\eta dx = \langle z(t),\eta\rangle_{(W^{1,q'})^*,W^{1,q'}}$$

for a. e. $t \in [0,T]$.

By an analogous reasoning as above, we obtain the existence of the distributional derivative $k' \in L^1(0,T;(W^{1,q'}(\Omega))^*)$ (thus $k \in C([0,T];(W^{1,q'}(\Omega))^*)$) and the function relation in (2.12) holds. In the same way as above, from (2.6) we conclude that $\langle k(0),\eta\rangle_{(W^{1,q'})^*,W^{1,q'}} = \int_\Omega k_0\eta dx$ for all $\eta \in W^{1,q'}(\Omega)$.

2.3 An existence theorem

The main result of our paper is the following

Theorem *Let $u_0 \in L^\infty(\Omega)$ and let $k_0 \in L^1(\Omega)$, $k_0 \ge 0$ a. e. in Ω. Then there exists a pair (u, k) such that*

$$u \in C_w([0,T]; L^2(\Omega)) \cap L^2(0,T; W_0^{1,2}(\Omega)), \quad u' \in L^2(0,T; W^{-1,4/3}(\Omega)), \tag{2.15}$$

$$\left.\begin{aligned} & k \in L^\infty(0,T; L^1(\Omega)) \cap \Big(\bigcap_{1<\rho<\frac{16}{7}} L^\rho(Q_T) \Big), \quad k \ge 0 \quad a.\,e.\ in \quad Q_T, \\ & \nabla k \in \bigcap_{1<q<\frac{4}{3}} [L^q(Q_T)]^2, \quad \delta\mu \int_{Q_T} \frac{|\nabla k|^2}{(1+k)^{1+\delta}} \le c \quad \forall\, \delta \in]0,1[, \\ & k^{1/2}\nabla k \in \bigcap_{1<r<\frac{8}{7}} [L^r(Q_T)]^2 \quad (c = const\ independent\ of\ \delta\ and\ \mu) \end{aligned}\right\} \tag{2.16}$$

$$k \in \bigcap_{1<r<\frac{8}{7}} C([0,T]; W^{-1,r}(\Omega)), \quad k' \in \bigcap_{1<r<\frac{8}{7}} L^1(0,T; W^{-1,r}(\Omega)), \tag{2.17}$$

$$k^{1/4}\nabla u \in [L^2(Q_T)]^2 \tag{2.18}$$

and

$$\langle u'(t), \xi \rangle_{W^{-1,4/3}, W_0^{1,4}} + \int_\Omega k^{1/2}(t)\nabla u(t) \cdot \nabla\xi dx = 0 \quad for\ a.\,e.\ t \in [0,T], \quad \forall\, \xi \in W_0^{1,4}(\Omega), \tag{2.19}$$

$$\left.\begin{aligned} & for\ some\ 1 < s < \tfrac{8}{7}, \\ & \langle k'(t), \eta \rangle_{W^{-1,s}, W_0^{1,s'}} + \int_\Omega \Big(\mu + k^{1/2}(t)\Big)\nabla k(t) \cdot \nabla\eta dx = \\ & = \int_\Omega \Big(k^{1/2}(t)|\nabla u(t)|^2 - k^{3/2}(t)\Big)\eta dx \quad for\ a.\,e.\ t \in [0,T], \quad \forall\, \eta \in W_0^{1,s'}(\Omega) \end{aligned}\right\} \tag{2.20}$$

$$u(0) = u_0 \quad in \quad L^2(\Omega), \quad k(0) = k_0 \quad in \quad W^{-1,s}(\Omega). \tag{2.21}$$

In addition, the pair (u, k) satisfies,

$$\min\Big\{0, \operatorname*{ess\,inf}_\Omega u_0\Big\} \le u \le \max\Big\{0, \operatorname*{ess\,sup}_\Omega u_0\Big\} \quad a.\,e.\ in \quad Q_T, \tag{2.22}$$

$$\frac{1}{2}\int_\Omega u^2(x,t)dx + \int_{Q_t} k^{1/2}|\nabla u|^2 \le \frac{1}{2}\int_\Omega u_0^2(x)dx, \tag{2.23}$$

$$\int_\Omega \Big(\frac{1}{2}u^2(x,t) + k(x,t)\Big)dx + \int_{Q_t} k^{3/2} \le \int_\Omega \Big(\frac{1}{2}u_0^2(x) + k_0(x)\Big)dx, \tag{2.24}$$

$$\int_{Q_t} |\nabla u|^2 \le 2\int_\Omega k^{1/2}(x,t)dx + \int_{Q_t} k \tag{2.25}$$

for a. e. $t \in [0,T]$.

The pair (u, k) obtained in the theorem above, exhibits the phenomenon of turbulence as follows.

Corollary ($k > 0$ *a. e. on a set of positive measure*). *Let be (u, k) as in the Theorem. Suppose that*

$$\int_{Q_T} |\nabla u|^2 > 0.$$

Then there exists a set $Q^* \subset Q_T$ such that

$$\operatorname{mes} Q^* > 0, \quad k > 0 \quad a.\,e. \quad on\ Q^*.$$

Proof Define $\alpha_0 := \int_{Q_T} |\nabla u|^2$. Fix $t_* \in]0,T[$ such that

$$\int_{t_*}^{T} \int_{\Omega} |\nabla u|^2 \leq \frac{\alpha_0}{2}.$$

Integration of (2.25) over $[t_*, T]$ gives

$$\frac{\alpha_0}{2}(T - t_*) \leq \int_{t_*}^{T} \int_{\Omega} k^{1/2} + \int_{t_*}^{T} \Big(\int_{Q_t} k \Big) dt.$$

Thus,

$$\alpha_0 \leq \Big(\frac{4 \operatorname{mes} \Omega}{T - t_*} \int_{Q_T} k \Big)^{1/2} + 2 \int_{Q_T} k.$$

Whence the claim.

3. Existence of an approximate solution $(u_\varepsilon, k_\varepsilon)$

In this section, we modify model problem (1.12)–(1.15) at two points. Firstly, by a standard cut-off method, we bound the coefficients $\sqrt{k}$ which occur in the differential operators on the left hand side of (1.12) and (1.13). Secondly, by adding the term $-\varepsilon \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$ $(\varepsilon > 0, p > 4)$ to the left hand side of (1.13) we make the model problem coercive. The existence of a weak solution to this modified problem is then easily proved by methods of abstract evolution equations. The proof of our existence theorem for weak solutions to (1.12)–(1.15) is then carried out by the passage to the limit $\varepsilon \to 0$.

The existence of a weak solution of problems of the type (1.1)–(1.3) with uniformly bounded coefficients in place of $(\nu + \ell\sqrt{k})$ and $(\mu + \ell\sqrt{k})$ in (1.2) and (1.3), respectively, has been proved in [11], [12]. The weak solution obtained in these works, however, does verify the scalar equation of type (1.3) with a defect measure.

To begin with, we define

$$[\tau]_\varepsilon : \min\Big\{\frac{1}{\varepsilon}, \tau\Big\}, \quad \varepsilon > 0, \quad 0 \leq \tau < +\infty.$$

Fix any $p > 4$. Given $\varepsilon > 0$, we consider the following problem: find a pair of functions $(u_\varepsilon, k_\varepsilon)$ such that $k_\varepsilon \geq 0$ in Q_T, and

$$\frac{\partial u_\varepsilon}{\partial t} - \operatorname{div}\Big(\big((\varepsilon + [k_\varepsilon]_\varepsilon)^{1/2} + \varepsilon|\nabla u_\varepsilon|^{p-2}\big)\nabla u_\varepsilon\Big) = 0 \quad \text{in} \quad Q_T, \tag{3.1}$$

$$\frac{\partial k_\varepsilon}{\partial t} - \operatorname{div}\Big(\big(\mu + (\varepsilon + [k_\varepsilon]_\varepsilon)^{1/2}\big)\nabla k_\varepsilon\Big) + \varepsilon k_\varepsilon + k_\varepsilon^{3/2} = (\varepsilon + [k_\varepsilon]_\varepsilon)^{1/2}|\nabla u_\varepsilon|^2 \text{ in } Q_T, \tag{3.2}$$

$$u_\varepsilon = 0, \quad \frac{\partial k_\varepsilon}{\partial \boldsymbol{n}'} = 0 \quad \text{on} \quad \partial\Omega \times [0,T], \tag{3.3}$$

$$u_\varepsilon = u_0, \quad k_\varepsilon = k_0 \quad \text{on} \quad \Omega \times \{0\}, \tag{3.4}$$

where $\boldsymbol{n}'$ denotes the unit normal to $\partial\Omega$. Formally, problem (3.1)–(3.4) turns into (1.12)–(1.15) when $\varepsilon \to 0$. We now prove the existence of a weak solution of (3.1)–(3.4).

Proposition 2 (*existence of an approximate solution*) *Let $u_0 \in W^{1,p}(\Omega)$ and $k_0 \in W^{1,2}(\Omega)$, $k_0 \geq 0$ a. e. in Ω. Then, for every $\varepsilon > 0$ there exists a pair $(u_\varepsilon, k_\varepsilon)$ such that*

$$u_\varepsilon \in C([0,T]; L^2(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega)), \quad u_\varepsilon' \in L^{p'}(0,T; W^{-1,p'}(\Omega)), \tag{3.5}$$

$$k_\varepsilon \in C([0,T]; L^2(\Omega)) \cap L^p(0,T; W^{1,2}(\Omega)), \quad k_\varepsilon' \in L^2(0,T; (W^{1,2}(\Omega))^*), \tag{3.6}$$

$$\min\Big\{0, \min_{\overline{\Omega}} u_0\Big\} \leq u_\varepsilon \leq \max\Big\{0, \max_{\overline{\Omega}} u_0\Big\} \text{ a. e. in } Q_T, \quad k_\varepsilon \geq 0 \text{ a. e. in } Q_T \tag{3.7}$$

and

$$\left.\begin{aligned}&\langle u_\varepsilon'(t),v\rangle_{W^{-1,p'},W_0^{1,p}}+\int_\Omega\Big((\varepsilon+[k_\varepsilon(t)]_\varepsilon)^{1/2}+\varepsilon|\nabla u_\varepsilon(t)|^{p-2}\Big)\nabla u_\varepsilon(t)\cdot\nabla v\,dx=0\\&\textit{for a. e. } t\in[0,T],\quad\forall\, v\in W_0^{1,p}(\Omega),\end{aligned}\right\}\tag{3.8}$$

$$\left.\begin{aligned}&\langle k_\varepsilon'(t),\varphi\rangle_{(W^{1,2})^*,W^{1,2}}+\int_\Omega\Big(\mu+(\varepsilon+[k_\varepsilon(t)]_\varepsilon)^{1/2}\Big)\nabla k_\varepsilon(t)\cdot\nabla\varphi\,dx\\&+\varepsilon\int_\Omega k_\varepsilon(t)\varphi\,dx+\int_\Omega k_\varepsilon^{3/2}(t)\varphi\,dx=\int_\Omega\Big(\varepsilon+[k_\varepsilon(t)]_\varepsilon\Big)^{1/2}|\nabla u_\varepsilon(t)|^2\varphi\,dx\\&\textit{for a. e. } t\in[0,T],\quad\forall\,\varphi\in W^{1,2}(\Omega),\end{aligned}\right\}\tag{3.9}$$

$$u_\varepsilon(0)=u_0\quad\textit{in}\quad L^2(\Omega),\qquad k_\varepsilon(0)=k_0\quad\textit{in}\quad L^2(\Omega).\tag{3.10}$$

Before turning to the proof of Proposition 2 we introduce some notations. Put

$$\mathcal{V}=L^p(0,T;W_0^{1,p}(\Omega))\times L^2(0,T;W^{1,2}(\Omega)).$$

By $(u,k),(v,\varphi),\dots$ we denote the elements of $\mathcal{V}$. The space $\mathcal{V}$ is reflexive with respect to the norm

$$\|(u,k)\|_{\mathcal{V}}:=\Big(\|u\|^2_{L^p(W_0^{1,p})}+\|k\|^2_{L^2(W^{1,2})}\Big)^{1/2}.$$

The dual space $\mathcal{V}^*$ is linearly isometric to $L^{p'}(0,T;W^{-1,p'}(\Omega))\times L^2(0,T;(W^{1,2}(\Omega))^*)$. Identifying this space with $\mathcal{V}^*$, we obtain, for all $(u^*,k^*)\in\mathcal{V}^*$ and all $(v,\varphi)\in\mathcal{V}$,

$$\begin{aligned}&\|(u^*,k^*)\|_{\mathcal{V}^*}=\Big(\|u^*\|^2_{L^{p'}(W^{-1,p'})}+\|k^*\|^2_{L^2((W^{1,2})^*)}\Big)^{1/2},\\&\langle(u^*,k^*),(v,\varphi)\rangle_{\mathcal{V}^*,\mathcal{V}}=\langle u^*,v\rangle_{L^{p'}(W^{-1,p'}),L^p(W_0^{1,p})}+\langle k^*,\varphi\rangle_{L^2((W^{1,2})^*),L^2(W^{1,2})}.\end{aligned}$$

Next, we define

$$\begin{aligned}D(\mathcal{L}):=\Big\{&(u,k)\in\mathcal{V};\exists\,(u'k')\in L^{p'}(0,T;W^{-1,p'}(\Omega))\times L^2(0,T;(W^{1,2}(\Omega))^*),\\&u(0)=0,\ k(0)=0\Big\},\\\mathcal{L}(u,k):=&\,(u',k'),\quad(u,k)\in D(\mathcal{L}).\end{aligned}$$

We furnish $D(\mathcal{L})$ with the usual graph norm

$$\|(u,k)\|_{D(\mathcal{L})}:=\|(u,k)\|_{\mathcal{V}}+\|\mathcal{L}(u,k)\|_{\mathcal{V}^*}.$$

The following results are well-known:

$$\begin{aligned}D(\mathcal{L})\subset\mathcal{V}&\quad\textit{continuously, densely}\\\mathcal{L}:D(\mathcal{L})\to\mathcal{V}^*&\quad\textit{is a linear, maximal monotone operator}\end{aligned}$$

(see, e. g., [10; Chap. 3]).

Proof of Proposition 2 For $\tau\in\mathbb{R}$, define $\tau^+:=\max\{\tau,0\}$, $\tau^-:=\min\{\tau,0\}$. Given $(u,k)\in D(\mathcal{L})$, put

$$\widehat{u}=u+u_0,\quad\widehat{k}=k+k_0.$$

Then, for $\varepsilon > 0$ we define a mapping $\mathcal{A}_\varepsilon : D(\mathcal{L}) \to \mathcal{V}^*$ by

$$\langle \mathcal{A}_\varepsilon(u,k),(v,\varphi)\rangle_{\mathcal{V}^*,\mathcal{V}} = \int_{Q_T} \Big((\varepsilon + [\widehat{k^+}]_\varepsilon)^{1/2} + \varepsilon|\nabla\widehat{u}|^{p-2}\Big)\nabla\widehat{u}\cdot\nabla v +$$
$$+ \int_{Q_T} \Big(\mu + (\varepsilon + [\widehat{k^+}]_\varepsilon)^{1/2}\Big)\nabla\widehat{k}\cdot\nabla\varphi dx + \int_{Q_T} \Big(\varepsilon\widehat{k} + \widehat{k^+}|\widehat{k}|^{1/2}\Big)\varphi - \int_{Q_T} (\varepsilon + [\widehat{k^+}]_\varepsilon)^{1/2}|\nabla\widehat{u}|^2\varphi,$$
$$(u,k) \in D(\mathcal{L}), \quad (v,k) \in \mathcal{V}.$$

We are now going to show that the mapping $\mathcal{A}_\varepsilon$ verifies the conditions (i), (ii) and (iii) below.

(i) *$\mathcal{A}_\varepsilon$ maps bounded sets in $D(\mathcal{L})$ into bounded sets in $\mathcal{V}^*$.* More precisely, for all $(u,k) \in \mathcal{V}$,

$$\|\mathcal{A}_\varepsilon(u,k)\|_{\mathcal{V}^*} \le \Psi_\varepsilon\Big(\|(u,k)\|_{\mathcal{V}}\Big) + \frac{1}{2}\|\mathcal{L}(u,k)\|_{\mathcal{V}^*} \tag{3.11}$$

where $\Psi_\varepsilon : [0,+\infty[\to]0,+\infty[$ is a non-decreasing function that is bounded on bounded intervals of $[0,+\infty[$, and $\Psi_\varepsilon(\tau) \to +\infty$ as $\varepsilon \to 0$, $\tau \in]0,+\infty[$.

To see this, it is evidently enough to observe that

$$\Big|\int_{Q_T} \widehat{k^+}|\widehat{k}|^{1/2}\varphi\Big| \le c\|\widehat{k}\|^{3/2}_{C([0,T];L^2)} \int_0^T \|\varphi(t)\|_{W^{1,2}}dt$$ [5]
$$\le \Big(c\|\widehat{k}\|^3_{L^2(W^{1,2})} + \frac{1}{2}\|k'\|_{L^2((W^{1,2})^*)}\Big)\|\varphi\|_{L^2(W^{1,2})}.$$

(by (2.9) with $X = W^{1,2}(\Omega)$, $H = L^2(\Omega)$), and

$$\Big|\int_{Q_T} (\varepsilon + [\widehat{k^+}]_\varepsilon)^{1/2}|\nabla u|^2\varphi\Big| \le \Big(\varepsilon + \frac{1}{\varepsilon}\Big)^{1/2}\|\nabla u\|^2_{[L^4(Q_T)]^2}\|\varphi\|_{L^2(W^{1,2})}.$$

(ii) *$\mathcal{A}_\varepsilon$ is coercive, i. e.*

$$\frac{\langle A_\varepsilon(u,k),(u,k)\rangle_{\mathcal{V}^*,\mathcal{V}}}{\|(u,k)\|_{\mathcal{V}}} \to +\infty \quad \textit{as} \quad (u,k) \in D(\mathcal{L}), \quad \|(u,k)\|_{\mathcal{V}} \to +\infty. \tag{3.12}$$

Indeed, by the definition of $\mathcal{A}_\varepsilon$,

$$\langle \mathcal{A}_\varepsilon(u,k),(u,k)\rangle_{\mathcal{V}^*,\mathcal{V}} =$$
$$= \int_{Q_T} \Big((\varepsilon + [\widehat{k^+}]_\varepsilon)^{1/2} + \varepsilon|\nabla\widehat{u}|^{p-2}\Big)\nabla\widehat{u}\cdot\nabla u + \int_{Q_T} \Big(\mu + (\varepsilon + [k^+]_\varepsilon)^{1/2}\Big)\nabla\widehat{k}\cdot\nabla k$$
$$+ \int_{Q_T} \Big(\varepsilon\widehat{k} + \widehat{k^+}|\widehat{k}|^{1/2}\Big)k - \int_{Q_T} (\varepsilon + [\widehat{k^+}]_\varepsilon)^{1/2}|\nabla\widehat{u}|^2 k.$$

It suffices to estimate the third and the fourth integral from below. We have

$$\int_{Q_T} (\varepsilon\widehat{k} + \widehat{k^+}|\widehat{k}|^{1/2})k \ge \varepsilon\int_{Q_T} k^2 + \varepsilon\int_{Q_T} kk_0 - \int_{Q_T} \widehat{k}^{3/2}k_0$$
$$\ge \frac{3\varepsilon}{4}\int_{Q_T} k^2 - c(\varepsilon)$$

and

$$-\int_{Q_T} (\varepsilon + [\widehat{k}]_\varepsilon)^{1/2}|\nabla\widehat{u}|^2 k \ge -\frac{\varepsilon}{4}\int_{Q_T} k^2 - \frac{\varepsilon}{2}\int_{Q_T} |\nabla\widehat{u}|^p - c(\varepsilon),$$

where the constants $c(\varepsilon)$ depend on $\|k_0\|_{W^{1,2}}$ (recall $p > 4$). Whence (3.12).

[5] Without any further reference, in what follows we denote by c constants which may change their numerical value from line to line.

(iii) $\mathcal{A}_\varepsilon$ *is pseudo-monotone* (with respect to weakly convergent sequences in $D(\mathcal{L})$), i. e. *for every sequence* $((u_j, k_j)) \subset D(\mathcal{L})$ *such that*

$$(u_j, k_j) \to (u, k) \text{ weakly in } \mathcal{V},\ \mathcal{L}(u_j, k_j) \to \mathcal{L}(u, k) \text{ weakly in } \mathcal{V}^*, \text{ and}$$

$$\limsup \langle A_\varepsilon(u_j, k_j), (u_j, k_j) - (u, k)\rangle_{\mathcal{V}^*,\mathcal{V}} \le 0 \tag{3.13}$$

there exists a subsequence (not relabelled) such that

$$\left.\begin{aligned} &\liminf \langle A_\varepsilon(u_j, k_j), (u_j, k_j) - (v, \varphi)\rangle_{\mathcal{V}^*,\mathcal{V}} \ge \\ &\ge \langle A_\varepsilon(u, k), (u, k) - (v, \varphi)\rangle_{\mathcal{V}^*,\mathcal{V}} \quad \forall\, (v, \varphi) \in \mathcal{V}. \end{aligned}\right\} \tag{3.14}$$

Before turning to the proof of (3.14) we notice that the convergence of the sequence $((k_j))$ implies the existence of a subsequence (not relabelled) such that

$$k_j \to k \quad \textit{strongly in } L^2(Q_T) \quad \textit{and} \quad \textit{a. e. in } Q_T \quad \textit{as} \quad j \to +\infty \tag{3.15}$$

(see [10; Chap. 1, Thm. 5.1], [22; Cor. 4]). Next, given $r > 2$, there holds

$$(|\xi|^{r-2}\xi - |\eta|^{r-2}\eta) \cdot (\xi - \eta) \ge \alpha_0 |\xi - \eta|^r \quad \forall\, \xi, \eta \in \mathbb{R}^n, \tag{3.16}$$

where $\alpha_0 = \text{const} > 0$ depends on r and n only. We obtain

$$\begin{aligned} &\alpha_0 \varepsilon \int_{Q_T} |\nabla(u_j - u)|^p \le \\ &\le \int_{Q_T} (\varepsilon + [\widehat{k^+}]_\varepsilon)^{1/2} |\nabla(\widehat{u}_j - \widehat{u})|^2 + \varepsilon \int_{Q_T} \Big(|\nabla\widehat{u}_j|^{p-2}\nabla\widehat{u}_j - |\nabla\widehat{u}|^{p-2}\nabla\widehat{u}\Big) \cdot \nabla(\widehat{u}_j - \widehat{u}) \\ &\quad + \int_{Q_T} \Big(\mu + (\varepsilon + [\widehat{k^+}]_\varepsilon)^{1/2}\Big) |\nabla(\widehat{k}_j - \widehat{k})|^2 \\ &= \int_{Q_T} (\varepsilon + [\widehat{k^+}]_\varepsilon)^{1/2} \nabla\widehat{u}_j \cdot \nabla(u_j - u) + \int_{Q_T} \Big(\mu + (\varepsilon + [\widehat{k^+}]_\varepsilon)^{1/2}\Big) \nabla\widehat{k}_j \cdot \nabla(k_j - k) + \mathcal{B}_{\varepsilon,j} \\ &= \langle \mathcal{A}_\varepsilon(u_j, k_j), (u_j, k_j) - (u, k)\rangle_{\mathcal{V}^*,\mathcal{V}} + \mathcal{B}_{\varepsilon,j} + C_{\varepsilon,j} \end{aligned} \tag{3.17}$$

(recall $u_j - u = \widehat{u}_j - \widehat{u}$, $k_j - k = \widehat{k}_j - \widehat{k}$). Observing that $\nabla\widehat{u}_j \to \nabla\widehat{u}$ weakly in $[L^p(Q_T)]^2$, $\nabla\widehat{k}_j \to \nabla\widehat{k}$ weakly in $[L^2(Q_T)]^2$ as $j \to +\infty$, and (3.15) we easily obtain

$$\lim \mathcal{B}_{\varepsilon,j} = \lim C_{\varepsilon,j} = 0.$$

Thus, from (3.13) and (3.17) it follows

$$\alpha_0 \varepsilon \limsup \int_{Q_T} |\nabla(u_j - u)|^p \le \limsup \Big(\langle \mathcal{A}_\varepsilon(u_j, k_j), (u_j, k_j) - (u, k)\rangle_{\mathcal{V}^*,\mathcal{V}} + \mathcal{B}_{\varepsilon,j} + C_{\varepsilon,j}\Big) \le 0.$$

Hence, by going to a subsequence if necessary,

$$\nabla u_j \to \nabla u \quad \text{strongly in} \quad [L^p(Q_T)]^2 \quad \text{and a. e. in} \quad Q_T \quad \text{as} \quad j \to +\infty. \tag{3.18}$$

We are now in a position to prove (3.14). Let $(v, \varphi) \in \mathcal{V}$. To form the lim inf in (3.14), we firstly notice that from (3.17) it follows

$$\lim \int_{Q_T} \Big(|\nabla\widehat{u}_j|^{p-2}\nabla u_j - |\nabla\widehat{u}|^{p-2}\nabla\widehat{u}\Big) \cdot \nabla(\widehat{u}_j - \widehat{u}) = 0.$$

Thus, by Minty's trick,

$$\liminf \int_{Q_T} |\nabla\widehat{u}_j|^{p-2}\nabla\widehat{u}_j \cdot \nabla(u_j - v) \geq \int_{Q_T} |\nabla\widehat{u}|^{p-2}\nabla\widehat{u} \cdot \nabla(u - v).$$

(see, e. g., [10; Chap. 2, Prop. 2.5]). Secondly, observing that

$$(\varepsilon + [\widehat{k}_j^+]_\varepsilon)^{1/4}\nabla\widehat{u}_j \to (\varepsilon + [\widehat{k}^+]_\varepsilon)^{1/4}\nabla\widehat{u} \quad \text{weakly in} \quad [L^2(Q_T)]^2,$$
$$\Big(\mu + (\varepsilon + [\widehat{k}_j^+]_\varepsilon)^{1/2}\Big)^{1/2}\nabla\widehat{k}_j \to \Big(\mu + (\varepsilon + [\widehat{k}^+]_\varepsilon)^{1/2}\Big)^{1/2}\nabla\widehat{k} \quad \text{weakly in} \quad [L^2(Q_T)]^2$$

as $j \to +\infty$, we find

$$\liminf \int_{Q_T} (\varepsilon + [\widehat{k}_j^+]_\varepsilon)^{1/2}\nabla\widehat{u}_j \cdot \nabla(u_j - v) \geq \int_{Q_T} (\varepsilon + [\widehat{k}^+]_\varepsilon)^{1/2}\nabla\widehat{u} \cdot \nabla(u - v),$$
$$\liminf \int_{Q_T} \Big(\mu + (\varepsilon + [\widehat{k}_j^+]_\varepsilon)^{1/2}\Big)\nabla\widehat{k}_j \cdot \nabla(k_j - \varphi) \geq \int_{Q_T} \Big(\mu + (\varepsilon + [\widehat{k}^+]_\varepsilon)^{1/2}\Big)\nabla\widehat{k} \cdot \nabla(k - \varphi).$$

Thirdly, taking into account (3.15) and (3.18) we obtain by routine arguments

$$\lim \int_{Q_T} \Big(\varepsilon\widehat{k}_j + \widehat{k}_j^+|\widehat{k}_j|^{1/2}\Big)(k_j - \varphi) = \int_{Q_T} \Big(\varepsilon\widehat{k} + \widehat{k}^+|\widehat{k}|^{1/2}\Big)(k - \varphi),$$
$$\lim \int_{Q_T} (\varepsilon + [\widehat{k}_j^+]_\varepsilon)^{1/2}|\nabla\widehat{u}_j|^2(k_j - \varphi) = \int_{Q_T} (\varepsilon + [\widehat{k}^+]_\varepsilon)^{1/2}|\nabla\widehat{u}|^2(k - \varphi).$$

Whence (3.14).

The mapping $\mathcal{A}_\varepsilon$ thus verifies the assumptions of [10; Chap. 3, Thm. 1.2]. Hence, for every $\varepsilon > 0$, there exists $(\widetilde{u}_\varepsilon, \widetilde{k}_\varepsilon) \in D(\mathcal{L})$ such that

$$\mathcal{L}(\widetilde{u}_\varepsilon, \widetilde{k}_\varepsilon) + \mathcal{A}_\varepsilon(\widetilde{u}_\varepsilon, \widetilde{k}_\varepsilon) = (0, 0) \quad \text{in} \quad \mathcal{V}^* \tag{3.19}$$

We define $u_\varepsilon := \widetilde{u}_\varepsilon + u_0$, $\quad k_\varepsilon := \widetilde{k}_\varepsilon + k_0$. Then (3.19) is equivalent to

$$\left.\begin{aligned} &\langle u_\varepsilon'(t), v\rangle_{W^{-1,p'},W_0^{1,p}} + \int_\Omega \Big((\varepsilon + [k_\varepsilon^+(t)]_\varepsilon)^{1/2} + \varepsilon|\nabla u_\varepsilon(t)|^{p-2}\Big)\nabla u_\varepsilon(t) \cdot \nabla v dx = 0 \\ &\text{for a. e. } t \in [0, T], \quad \forall\, v \in W_0^{1,p}(\Omega), \end{aligned}\right\} \tag{3.20}$$

$$\left.\begin{aligned} &\langle k_\varepsilon'(t), \varphi\rangle_{(W^{1,2})^*,W^{1,2}} + \int_\Omega \Big(\mu + (\varepsilon + [k_\varepsilon^+(t)]_\varepsilon)^{1/2}\Big)\nabla k_\varepsilon(t) \cdot \nabla\varphi dx \\ &+ \int_\Omega \Big(\varepsilon k_\varepsilon(t) + k_\varepsilon^+(t)|k_\varepsilon(t)|^{1/2}\Big)\varphi dx = \int_\Omega (\varepsilon + [k_\varepsilon^+(t)]_\varepsilon)^{1/2}|\nabla u_\varepsilon(t)|^2\varphi dx \\ &\text{for a. e. } t \in [0, T], \quad \forall\, \varphi \in W^{1,2}(\Omega). \end{aligned}\right\} \tag{3.21}$$

To prove the bounds on u_ε (cf. (3.7)), put

$$\lambda_* = \min\Big\{0, \operatorname*{ess\,inf}_\Omega u_0\Big\}, \quad \lambda^* = \max\Big\{0, \operatorname*{ess\,sup}_\Omega u_0\Big\}$$

Then, for a. e. $t \in [0, T]$, $v = \Big(u_\varepsilon(\cdot, t) - \lambda_*\Big)^- \in W_0^{1,p}(\Omega)$. From (3.19) it follows that

$$\langle u_\varepsilon'(t), (u_\varepsilon(t) - \lambda_*)^-\rangle_{W^{-1,p'},W_0^{1,p}} \leq 0 \quad \text{for a. e. } t \in [0, T]$$

Thus $u_\varepsilon - \lambda_* \geq 0$ a. e. in Q_T. Analogously, $u_\varepsilon - \lambda^* \leq 0$ a. e. in Q_T. Finally, the function $\varphi = k_\varepsilon^-(\cdot, t)$ being admissible in (3.21) we find

$$\langle k_\varepsilon'(t), k_\varepsilon^-(t)\rangle_{(W^{1,2})^*,W^{1,2}} \leq 0 \quad \text{for a. e. } t \in [0, T],$$

and therefore $k_\varepsilon \geq 0$ a. e. in Q_T.

The pair $(u_\varepsilon, k_\varepsilon)$ verifies (3.5)–(3.10) of Proposition 2.

4. Proof of the existence theorem

4.1 A-priori estimates

Let $u_0 \in L^\infty(\Omega)$ and let $k_0 \in L^1(\Omega)$, $k_0 \geq 0$ a. e. in Ω. Fix $p > 4$. For every $\varepsilon > 0$, there exists $u_{\varepsilon,0} \in W_0^{1,p}(\Omega)$ and $k_{0,\varepsilon} \in W^{1,2}(\Omega)$ such that

$$\min\Big\{0, \operatorname*{ess\,inf}_\Omega u_0\Big\} \leq u_{\varepsilon,0} \leq \max\Big\{0, \operatorname*{ess\,sup}_\Omega u_0\Big\}, \quad k_{0,\varepsilon} \geq 0 \quad \text{a. e. in } \Omega,$$

$$u_{0,\varepsilon} \to u_0 \quad \text{in} \quad L^2(\Omega), \quad k_{0,\varepsilon} \to k_0 \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad \varepsilon \to 0.$$

Then from Proposition 2 it follows that there exists a pair $(u_\varepsilon, k_\varepsilon)$ that verifies (3.5)–(3.10) with $u_{0,\varepsilon}$ and $k_{0,\varepsilon}$ in place of u_0 and k_0, respectively.

For notational simplicity, throughout the present section we omit the index ε at u_ε and k_ε. Without any further reference, in all of Section 4.1, let $0 < \varepsilon \leq 1$.

(i) We insert $v = u(\cdot, t)$ $(0 \leq t \leq T)$ into (3.8) and integrate over $[0, t]$. It follows

$$\max_{t\in[0,T]} \int_\Omega u^2(x,t)dx + \int_{Q_T} \Big((\varepsilon + [k]_\varepsilon)^{1/2} + \varepsilon|\nabla u|^{p-2}\Big)|\nabla u|^2 \leq \frac{3}{2}\int_\Omega u_0^2(x)dx. \tag{4.1}$$

(ii) Let $0 < \delta < 1$. We define

$$\phi_1(\xi) = \phi_{1;\varepsilon,\delta}(\xi) = \int_0^\xi \Big(1 - \frac{1}{(\varepsilon + s)^\delta}\Big)ds, \quad 0 \leq \xi < +\infty.$$

Observing that $\phi_1 \in C^2([0, +\infty[)$ with ϕ_1' uniformly bounded on $[0, +\infty[$ one obtains by the aid of the chain rule for Sobolev functions

$$\begin{aligned}\int_0^t \langle k'(s), \phi_1'(k(s))\rangle_{(W^{1,2})^*, W^{1,2}} ds &= \int_0^t \frac{\partial}{\partial s}\Big(\int_\Omega \phi_1(k(x,s))dx\Big)ds = \\ &= \int_\Omega \phi_1(k(x,t))dx - \int_\Omega \phi_1(k_0(x))dx \quad \forall\, t \in [0,T].\end{aligned}$$

Take $\varphi = \phi_1'(k(\cdot,t))$ in (3.9). Integration over $[0, t]$ gives

$$\begin{aligned}&\int_\Omega \phi_1(k(x,t))dx + \delta\int_{Q_t}\Big(\mu + (\varepsilon + [k]_\varepsilon)^{1/2}\Big)\frac{|\nabla k|^2}{(\varepsilon + k)^{1+\delta}} + \int_{Q_t}(\varepsilon k + k^{3/2})\Big(1 - \frac{1}{(\varepsilon + k)^\delta}\Big) = \\ &= \int_{Q_t}(\varepsilon + [k]_\varepsilon)^{1/2}|\nabla u|^2\Big(1 - \frac{1}{(\varepsilon + k)^\delta}\Big) + \int_\Omega \phi_1(k_0(x))dx \\ &\leq \frac{3}{2}\int_\Omega u_0^2(x)dx \qquad \text{(by (4.1)).}\end{aligned} \tag{4.2}$$

With the help of the elementary inequalities

$$\xi \geq \phi_1(\xi) \geq \frac{\xi}{2} - c_1(\delta), \quad \xi^a\Big(1 - \frac{1}{(\varepsilon + \xi)^\delta}\Big) \geq \frac{\xi^a}{2} - c_2(q,\delta) \quad \forall\, \xi \in [0, +\infty[\quad \Big(a = 1, a = \frac{3}{2}\Big)$$

($c_1(\delta) = \text{const} > 0$, $c_2(a,\delta) = \text{const} > 0$ independent of ε) we infer from (4.2)

$$\|k\|_{L^\infty(L^1)} + \|k\|^{3/2}_{L^{3/2}(Q_T)} + \delta\mu\int_{Q_T}\frac{|\nabla k|^2}{(1 + k)^{1+\delta}} \leq c. \tag{4.3}$$

(iii) From (4.1) and (4.3) one easily deduces an estimate on $\|u'\|_{L^{p'}(W^{-1,p'})}$. Indeed, (3.8) implies, for a. e. $t \in [0, T]$ and all $v \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}&\Big|\langle u'(t), v\rangle_{W^{-1,p'}, W_0^{1,p}}\Big| \leq \\ &\leq \Big\{(\text{mes}\,\Omega)^{p/(p-4)}\|\varepsilon + k\|^{1/4}_{L^\infty(L^1)}\Big(\int_\Omega(\varepsilon + [k(t)]_\varepsilon)^{1/2}|\nabla u(t)|^2dx\Big)^{1/2} \\ &\qquad + \varepsilon\Big(\int_\Omega|\nabla u(t)|^pdx\Big)^{(p-1)/p}\Big\}\|\nabla v\|_{L^p}.\end{aligned}$$

By (4.1) and (4.3), the function in brackets $\{\dots\}$ is uniformly bounded with respect to the norm in $L^{p'}(0,T)$. Thus

$$\|u'\|_{L^{p'}(W^{-1,p'})} \le c. \tag{4.4}$$

(iv) We deduce from (3.9) an estimate on $\int_{Q_T} |\nabla u|^2$ in terms of integral norms of k. To this end, we define

$$\phi_2(\xi) = \phi_{2;\varepsilon}(\xi) = \int_0^\xi \frac{ds}{(\varepsilon + [s]_\varepsilon)^{1/2}}, \quad 0 \le \xi < +\infty.$$

Clearly, $\phi_2 \in C^1([0,+\infty[)$ and $0 \le \phi_2'(\xi) \le \frac{1}{\varepsilon^{1/2}}$ for all $\xi \in [0,+\infty[$. As above,

$$\int_0^t \langle k'(t), \phi_2'(k(s))\rangle_{(W^{1,2})^*,W^{1,2}} ds = \int_\Omega \phi_2(k(x,t))dx - \int_\Omega \phi_2(k_0(x))dx \quad \forall\, t \in [0,T].$$

Next, since ϕ_2'' is continuous on $[0,+\infty[\setminus\left\{\frac{1}{\varepsilon}\right\}$, we have

$$\nabla\phi_2'(k(\cdot,t)) = -\frac{\nabla[k(\cdot,t)]_\varepsilon}{2(\varepsilon + [k(\cdot,t)]_\varepsilon)^{3/2}} \quad \text{a.e. in} \quad \Omega.$$

Thus, from (3.9) with $\varphi = \phi_2'(k(\cdot,t))$ therein it follows

$$\begin{aligned}\int_\Omega \phi_2(k_0(x))dx + \int_{Q_t} |\nabla u|^2 = \int_\Omega \phi_2(k(x,t))dx - \frac{1}{2}\int_{Q_t} \Big(\mu + (\varepsilon + [k]_\varepsilon)^{1/2}\Big)\frac{\nabla k \cdot \nabla [k]_\varepsilon}{(\varepsilon + [k]_\varepsilon)^{3/2}} \\ + \int_{Q_t} \frac{\varepsilon k + k^{3/2}}{(\varepsilon + [k]_\varepsilon)^{1/2}} \qquad \forall\, t \in [0,T].\end{aligned} \tag{4.5}$$

Clearly, the second term on the right hand side of (4.5) is ≤ 0, while the third term on the right hand side of (4.5) is easily estimated by

$$\int_{Q_t} \frac{\varepsilon k + k^{3/2}}{(\varepsilon + [k]_\varepsilon)^{1/2}} \le \varepsilon^{1/2}\int_{Q_t} k(1 + k^{1/2}) + \int_{Q_t} k.$$

The integrals in (4.5) which involve ϕ_2, can be estimated by observing the definition of $[\xi]_\varepsilon$ (with $\xi = k_0(x)$ and $\xi = k(x,t)$) as follows

$$\int_\Omega \phi_2(k_0(x))dx \ge 2\int_{\left\{k_0(x)\le\frac{1}{\varepsilon}\right\}} k_0^{1/2}(x)dx - 2\varepsilon^{1/2}\text{mes}\,\Omega$$

and

$$\begin{aligned}\int_\Omega \phi_2(k(x,t))dx &= 2\int_{\left\{k(x,t)\le\frac{1}{\varepsilon}\right\}} \Big[(\varepsilon + k(x,t))^{1/2} - \varepsilon^{1/2}\Big]dx \\ &\quad + 2\int_{\left\{k(x,t)>\frac{1}{\varepsilon}\right\}} \Big[\Big(\varepsilon + \frac{1}{\varepsilon}\Big)^{1/2} - \varepsilon^{1/2}\Big]dx + \frac{\varepsilon^{1/2}}{(1+\varepsilon^2)^{1/2}}\int_{\left\{k(x,t)>\frac{1}{\varepsilon}\right\}} \Big(k(x,t) - \frac{1}{\varepsilon}\Big)dx \\ &\le 2\int_\Omega k^{1/2}(x,t)dx + \varepsilon^{1/2}\int_\Omega k(x,t)dx\end{aligned}$$

for all $t \in [0,T]$. Thus, from (4.5) it follows

$$\begin{aligned}&2\int_{\left\{k(x,t)\le\frac{1}{\varepsilon}\right\}} k_0^{1/2}(x)dx - 2\varepsilon^{1/2}\text{mes}\,\Omega + \int_{Q_t} |\nabla u|^2 \\ &\quad \le 2\int_\Omega k^{1/2}(x,t)dx + \int_{Q_t} k + \varepsilon^{1/2}\Big(\int_\Omega k(x,t)dx + \int_{Q_t} k(1 + k^{1/2})\Big)\end{aligned} \tag{4.6}$$

for all $t \in [0,T]$. Hence, by (4.3), $\int_{Q_T} |\nabla u|^2$ is uniformly bounded by constant that does not depend on ε.

(v) We now deduce from (3.9) an estimate on $\int_{Q_T} |\nabla k|^q$, $\left(1 \le q < \frac{4}{3}\right)$. For the time being, let $1 \le q < n$ $(n = 2, 3, \ldots)$. Combining Hölder's inequality and Sobolev's embedding theorem we obtain, for all $w \in L^\infty(0,T;L^1(\Omega)) \cap L^q(0,T;W^{1,q}(\Omega))$,

$$\int_{Q_T} |w|^{(1+n)q/n} \le c\|w\|^{q/n}_{L^\infty(L^1)}\Big(\|w\|^q_{L^\infty(L^1)} + \|\nabla w\|^q_{[L^q(Q_T)]^n}\Big).$$

Thus, taking into account (4.3), we find, for any $1 \le q \le 2$,

$$\int_{Q_T} k^{3q/2} \le c\Big(1 + \int_{Q_T} |\nabla k|^q\Big) \tag{4.7}$$

(recall that $k \in L^2(0,T;W^{1,2}(\Omega))$, cf. Prop. 2).

To proceed, let $1 < q < \frac{4}{3}$ and define $\delta = \delta_q = \frac{4-3q}{2}$. Then $0 < \delta < 1$ and $\frac{(1+\delta)q}{2-q} = \frac{3q}{2}$. Hence, by (4.3) and (4.7),

$$\begin{aligned}\int_{Q_T} |\nabla k|^q &\le \Big(\int_{Q_T} \frac{|\nabla k|^2}{(1+k)^{1+\delta}}\Big)^{q/2}\Big(\int_{Q_T} (1+k)^{(1+\delta)q/(2-q)}\Big)^{(2-q)/2}\\ &\le c\Big(\frac{1}{\delta\mu}\Big)^{q/2}\Big(1 + \Big(\int_{Q_T} |\nabla k|^q\Big)^{(2-q)/2}\Big).\end{aligned}$$

Thus,

$$\int_{Q_T} |\nabla k|^q \le c \quad \forall\, 1 < q < \frac{4}{3} \quad \Big(c = c(q) \to +\infty \text{ as } q \to \frac{4}{3}\Big). \tag{4.8}$$

(vi) In order to prove that $[k]^{1/2}_\varepsilon \nabla k$ is uniformly bounded in $[L^r(Q_T)]^2$ for some $r > 1$, we consider the functions

$$\phi_3(\xi) = \phi_{3;\varepsilon}(\xi) = \int_0^\xi [s]^{1/2}_\varepsilon ds, \quad \phi_4(\xi) = \phi_{4;\varepsilon}(\xi) = \int_0^\xi \Big(1 - \frac{1}{(1+\phi_3(s))^\gamma}\Big)ds, \quad 0 \le \xi < +\infty,$$

where $0 < \gamma < 1$ will be specified below. Obviously, $\phi_4 \in C^2([0,+\infty[)$. The function $\varphi = \phi_4'(k(\cdot,t))$ is admissible in (3.9). Observing that

$$\nabla\varphi = \frac{\gamma[k(\cdot,t)]^{1/2}_\varepsilon}{(1+\phi_3(k(\cdot,t)))^{1+\gamma}}\nabla k(\cdot,t) \quad \text{a.e. in} \quad \Omega,$$

it follows

$$\begin{aligned}&\int_\Omega \phi_4(k(x,T))dx + \gamma\int_{Q_T} \Big(\mu + (\varepsilon + [k]_\varepsilon)^{1/2}\Big)\frac{[k]^{1/2}_\varepsilon|\nabla k|^2}{(1+\phi_3(k))^{1+\gamma}} + \int_{Q_T} (\varepsilon k + k^{3/2})\Big(1 - \frac{1}{(1+\phi_3(k))^\gamma}\Big)\\ &= \int_{Q_T} (\varepsilon + [k]_\varepsilon)^{1/2}|\nabla u|^2\Big(1 - \frac{1}{(1+\phi_3(k))^\gamma}\Big) + \int_\Omega \phi_4(k_0(x))dx\\ &\le \int_\Omega \Big(\frac{3}{2}u_0^2(x) + k_0(x)\Big)dx\end{aligned} \tag{4.9}$$

We notice that a test function of the type $\varphi = \phi_4'(k)$ has been used in [6].

To proceed, put $w = \phi_3(k)$. We have

$$\left.\begin{aligned}&\int_{Q_T} w^q \le c \qquad \forall\, 1 \le q < \frac{4}{3} \quad \text{[by (4.7) and (4.8)]},\\ &\gamma\int_{Q_T} \frac{|\nabla w|^2}{(1+w)^{1+\gamma}} \le c \quad \forall\, 0 < \gamma < 1 \quad \text{[by (4.9)]}.\end{aligned}\right\} \tag{4.10}$$

We take $1 \le r < \frac{8}{7}$ and $0 < \gamma < \frac{8-7r}{3r}$. Then $1 < \frac{(1+\gamma)r}{2-r} < \frac{4}{3}$. From (4.10) with $q = \frac{(1+\gamma)r}{2-r}$ we obtain by the same argument as (4.8)

$$\|w\|_{L^r(W^{1,r})} \le c \quad \forall\, 1 \le r < \frac{8}{7} \quad \Big(c = c(r) \to +\infty \quad \text{as} \quad r \to \frac{8}{7}\Big). \tag{4.11}$$

We show that (4.1) and (4.11) imply the estimate

$$\|w^{2/3}\|_{L^\rho(Q_T)} \le c \quad \forall\, 1 \le \rho < \frac{16}{7} \quad \Big(c = c(\rho) \to +\infty \quad \text{as} \quad \rho \to \frac{16}{7}\Big). \tag{4.12}$$

Indeed, put $z = w^{2/3}$. By (4.1), $\|z\|_{L^\infty(L^1)} \le c$. On the other hand, by Sobolev's embedding theorem and (4.11),

$$\int_0^T \|z\|^{3r/2}_{L^{3r/(2-r)}} dt = \int_0^T \|w\|^r_{L^{2r/(2-r)}} dt \le \int_0^T \|w\|^r_{W^{1,r}} dt \le c.$$

Thus, by interpolation,

$$\|z\|_{L^{2r}(L^{2r})} \le \|z\|^{3/4}_{L^{3r/2}(L^{3r/(2-r)})} \|z\|^{1/4}_{L^\infty(L^1)} \le c \quad \forall\, 1 \le r < \frac{8}{7}.$$

Whence, (4.12).

(vii) We finally prove an a-priori estimate on $\|k'\|_{L^1((W^{1,r'})^*)}$ $\big(1 < r < \frac{8}{7}\big)$. We insert $\varphi \in W^{1,r'}(\Omega)$ into (3.9). Taking into account that $\max_{\overline{\Omega}} |\varphi| \le c\|\varphi\|_{W^{1,r'}}$ it follows

$$\begin{aligned} &\Big|\langle k'(t), \varphi\rangle_{(W^{1,r'})^*, W^{1,r'}}\Big| = \Big|\langle k'(t), \varphi\rangle_{(W^{1,2})^*, W^{1,2}}\Big| \\ &\le \bigg\{\Big(\int_\Omega \Big[\big(\mu + (\varepsilon + [k(t)]_\varepsilon)^{1/2}\big)|\nabla k(t)|\Big]^r dx\Big)^{1/r} + \\ &\qquad + c\int_\Omega \Big(\varepsilon k(t) + k^{3/2}(t) + (\varepsilon + [k(t)]_\varepsilon)^{1/2}|\nabla u(t)|^2\Big) dx\bigg\}\|\varphi\|_{W^{1,r'}}. \end{aligned}$$

Estimates (4.1), (4.3), (4.8) and (4.11) show that the function in brackets $\{\dots\}$ is uniformly bounded in $L^1(Q_T)$ for all $0 < \varepsilon \le 1$. Thus,

$$\|k'\|_{L^1((W^{1,r'})^*)} \le c. \tag{4.13}$$

4.2 Passage to the limit $\varepsilon \to 0$

Firstly, from (4.1), (4.3) (combined with (4.6) and (4.4)) we obtain a subsequence of (u_ε) (not relabelled) and an element $h \in L^2(\Omega)$ such that

$$\left.\begin{aligned} &u_\varepsilon \to u \quad \text{weakly in } L^2(0,T; W^{1,2}_0(\Omega)) \text{ and weakly}* \text{ in } L^\infty(0,T; L^2(\Omega)) \\ &u'_\varepsilon \to u' \quad \text{weakly in } L^{p'}(0,T; W^{-1,p'}_0(\Omega)),\ u_\varepsilon(T) \to h \text{ weakly in } L^2(\Omega)) \end{aligned}\right\} \tag{4.14}$$

as $\varepsilon \to 0$. In addition, again by passing to a subsequence if necessary, we may assume that

$$u_\varepsilon \to u \quad \text{strongly in } L^2(Q_T) \text{ and a. e. in } Q_T \tag{4.15}$$

as $\varepsilon \to 0$ (see, e. g., [10; Chap. 1, Thm. 5.1], [22; Cor. 4]). The passage to the limit $\varepsilon \to 0$ in the inequality on u_ε in (3.7) gives the inequality on u in (2.22).

Secondly, to select an appropriate subsequence of (k_ε) we notice that $W^{1,q}(\Omega) \subset L^2(\Omega)$ $\big(1 < q < \frac{4}{3}\big)$ compactly and $L^2(\Omega) \cong (L^2(\Omega))^* \subset (W^{1,r'}(\Omega))^*$ $\big(1 < r < \frac{8}{7}\big)$ continuously. Then (4.3), (4.8) and (4.13) imply the existence of a subsequence of (k_ε) such that

$$k_\varepsilon \to u \quad \text{weakly in } L^q(0,T; W^{1,q}(\Omega)) \text{ strongly in } L^q(0,T; L^2(\Omega)) \text{ and a. e. in } Q_T \tag{4.16}$$

as $\varepsilon \to 0$ (see [22; Cor. 4]). Clearly, $k \geq 0$ a. e. in Q_T. With the help of these convergence properties we conclude from (4.1), (4.3), (4.11) and (4.12) by routine arguments that

$$\left.\begin{aligned} &(\varepsilon + [k_\varepsilon]_\varepsilon)^{1/4}\nabla u_\varepsilon \to k^{1/4}\nabla u \quad \text{weakly in } [L^2(Q_T)]^2,\\ &\frac{\nabla k\varepsilon}{(1+k_\varepsilon)^{(1+\delta)/2}} \to \frac{\nabla k}{(1+k)^{(1+\delta)/2}} \quad \text{weakly in } [L^2(Q_T)]^2,\\ &[k_\varepsilon]_\varepsilon^{1/2}\nabla k_\varepsilon \to k^{1/2}\nabla k \quad \text{weakly in } [L^r(Q_T)]^2 \ \left(1 < r < \tfrac{8}{7}\right),\\ &w_\varepsilon^{2/3} = (\phi_3(k_\varepsilon))^{2/3} \to \frac{2}{3}k \quad \text{weakly in } L^\rho(Q_T)\ \left(1 < \rho < \frac{16}{7}\right) \end{aligned}\right\} \tag{4.17}$$

as $\varepsilon \to 0$ (cf. part. (vi) of the a-priori estimates). To obtain $\nabla(k^{3/2}) = \frac{3}{2}k^{1/2}\nabla k$ we have made use of an elementary extension of the usual chain rule for Sobolev functions for the case $\phi(\xi) = \xi^{3/2}$ ($\xi \in [0, +\infty[$).

Next, from (3.8) and (3.9) we conclude that, for all $t \in [0, T]$,

$$\frac{1}{2}\int_\Omega u_\varepsilon^2(x,t)dx + \int_{Q_t}(\varepsilon + [k_\varepsilon]_\varepsilon)^{1/2}|\nabla u_\varepsilon|^2 \leq \frac{1}{2}\int_\Omega u_0^2(x)dx, \tag{4.18}$$

$$\int_\Omega \Big(\frac{1}{2}u_\varepsilon^2(x,t) + k_\varepsilon(x,t)\Big)dx + \int_{Q_t} k^{3/2} \leq \int_\Omega \Big(\frac{1}{2}u_0^2(x) + k_0(x)\Big)dx. \tag{4.19}$$

To pass to the limit $\varepsilon \to 0$ in (4.3), (4.6) and (4.18), (4.19), we notice as prototype the following elementary result:

$$\left.\begin{aligned} &\textit{Let } (f_m) \subset L^1(Q_T), (g_m) \subset L^p(Q_T)\ (1 < p < +\infty) \textit{ be sequences such that}\\ &f_m \geq 0,\ g_m \geq 0 \textit{ a. e. in } Q_T,\ \int_\Omega f_m(x,t)dx + \int_{Q_t} g_m^p \leq C_0 = \text{const} \textit{ for a. e. } t \in [0,T]\\ &(m = 1, 2, \ldots), \textit{ and}\\ &f_m \to f \textit{ weakly in } L^1(Q_T), \quad g_m \to g \textit{ weakly in } L^p(Q_T) \textit{ as } m \to +\infty.\\ &\textit{Then} \quad \int_\Omega f(x,t)dx + \int_{Q_t} g^p \leq C_0 \quad \textit{for a. e. } t \in [0,T]. \end{aligned}\right\} \tag{4.20}$$

Now, combining (4.16), (4.17) and (4.20) with (4.6), (4.18), (4.19) one easily deduces (2.16) and (2.23)–(2.25).

By (4.14) and (4.17), from (3.8) it follows (first for $v \in L^p(0, T; W_0^{1,p}(\Omega))$ and then by approximation) that

$$-\int_{Q_T} u\frac{\partial v}{\partial t} + \int_{Q_T} k^{1/2}\nabla u \cdot \nabla v = \int_\Omega u_0(x)v(x)dx$$

for all $v \in L^2(0, T; W_0^{1,4}(\Omega))$ with $\dfrac{\partial v}{\partial t} \in L^2(Q_T)$, $v(\cdot, T) = 0$ (cf. (2.5)). Moreover, there exist the distributional derivative $u' \in L^2(0, T; W^{-1,4/3}(\Omega))$, and $u(0) = u_0$ in $L^2(\Omega)$ (cf. Proposition 1, Section 2.2).

Next, we prove $h = u(T)$ in $L^2(\Omega)$ (cf. (4.14)). Indeed, for every $v \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}(h, v)_{L^2} - (u_0, v)_{L^2} &= \lim \int_0^T \langle u_\varepsilon'(t), v\rangle_{W^{-1,p'}, W_0^{1,p}} dt = \int_0^T \langle u'(t), v\rangle_{W^{-1,p'}, W_0^{1,p}} dt\\ &= (u(T), v)_{L^2} - (u_0, v)_{L^2}.\end{aligned}$$

Whence the claim.

It remains to carry out the passage to the limit $\varepsilon \to 0$ in (3.9), where we write

$$\int_0^T \langle k_\varepsilon', \varphi\rangle_{(W^{1,2})^*, W^{1,2}} dt = -\int_{Q_T} k_\varepsilon \frac{\partial \varphi}{\partial t}$$

with appropriate test functions φ (to be specified below). Here, the passage to the limit $\varepsilon \to 0$ of the L^1-term on the right hand side in (3.9) is the only crucial point.

To do this, let $\zeta \in C_0^1(\Omega)$, $\zeta \geq 0$ in Ω. By (4.17), $(\varepsilon + [k_\varepsilon]_\varepsilon)^{1/4}(\nabla u_\varepsilon)\zeta^{1/2} \to k^{1/4}(\nabla u)\zeta^{1/2}$ weakly in $[L^2(Q_T)]^2$ as $\varepsilon \to 0$ and thus

$$\int_{Q_T} k^{1/2}|\nabla u|^2 \zeta \leq \liminf \int_{Q_T} (\varepsilon + [k_\varepsilon]_\varepsilon)^{1/2} |\nabla u_\varepsilon|^2 \zeta. \tag{4.21}$$

On the other hand, the function $v = u_\varepsilon \zeta$ being admissible in (3.8), we find

$$\int_{Q_T} (\varepsilon + [k_\varepsilon]_\varepsilon)^{1/2} |\nabla u_\varepsilon|^2 \zeta \leq$$
$$\leq -\frac{1}{2}\int_\Omega u_\varepsilon^2(x,T)\zeta(x)dx + \frac{1}{2}\int_\Omega u_0^2(x)\zeta(x) - \int_{Q_T} \Big((\varepsilon + [k_\varepsilon]_\varepsilon)^{1/2} + \varepsilon|\nabla u_\varepsilon|^{p-2}\Big)(\nabla u_\varepsilon \cdot \nabla\zeta)u_\varepsilon.$$

Thus, by (4.14) (recall $h = u(T)$ in $L^2(\Omega)$) and (4.15),

$$\limsup \int_{Q_T} (\varepsilon + [k_\varepsilon]_\varepsilon)^{1/2} |\nabla u_\varepsilon|^2 \zeta \leq$$
$$\leq -\frac{1}{2}\int_\Omega u^2(x,T)\zeta(x)dx + \frac{1}{2}\int_\Omega u_0^2(x)dx - \int_{Q_T} k^{1/2}(\nabla u \cdot \nabla \zeta)u$$
$$= \int_{Q_T} k^{1/2}|\nabla u|^2\zeta \quad \text{[by the local energy equality (A.9)]}$$
$$\leq \liminf \int_{Q_T} (\varepsilon + [k_\varepsilon]_\varepsilon)^{1/2} |\nabla u_\varepsilon|^2 \zeta.$$

Hence, by (4.21),

$$\lim \int_{Q_T} (\varepsilon + [k_\varepsilon]_\varepsilon)^{1/2} |\nabla u_\varepsilon|^2 \zeta = \int_{Q_T} k^{1/2}|\nabla u_\varepsilon|^2 \zeta.$$

This equality continues to hold for all $\zeta \in W_0^{1,p}(\Omega)$, $\zeta \geq 0$ a. e. in Ω (recall $p > 4$). It follows

$$\lim \int_{Q_T} \Big|(\varepsilon + [k_\varepsilon]_\varepsilon)^{1/2} |\nabla u_\varepsilon|^2 z\alpha - k^{1/2}|\nabla u|^2 z\alpha\Big| = 0 \quad \forall\, z \in W_0^{1,p}(\Omega),\ \forall\, \alpha \in L^\infty(0,T). \tag{4.22}$$

Let $\varphi \in C([0,T]; W_0^{1,p}(\Omega))$. Then there exist $\varphi_m = \sum_{j=1}^m z_{m_j} t^j$ ($z_{m_j} \in W_0^{1,p}(\Omega)$; $m = 1, 2, \ldots$) such that $\varphi_m \to \varphi$ in $C([0,T]; W_0^{1,p}(\Omega))$. Thus, by (4.22),

$$\lim \int_{Q_T} (\varepsilon + [k_\varepsilon]_\varepsilon)^{1/2} |\nabla u_\varepsilon|^2 v = \int_{Q_T} k^{1/2}|\nabla u|^2 v.$$

From (3.9) it now follows that

$$-\int_{Q_T} k\frac{\partial\varphi}{\partial t} + \int_{Q_T} (\mu + k^{1/2})\nabla u \cdot \nabla\varphi = \int_\Omega k_0(x)\varphi(x,0)dx + \int_{Q_T} \Big(k^{1/2}|\nabla u|^2 - k^{3/2}\Big)\varphi$$

for all $\varphi \in C^1([0,T]; W_0^{1,s'}(\Omega))$ $\big(1 < s < \frac{8}{7}\big)$ such that $\dfrac{\partial\varphi}{\partial t} \in L^2(Q_T)$ and $\varphi(T) = 0$. Finally, by Proposition 1 (with $W_0^{1,s'}(\Omega)$ in place of $W^{1,q'}(\Omega)$ in (2.6)),

$$\exists\, k' \in L^1(0,T; W^{-1,s}(\Omega)), \quad k(0) = k_0 \quad \text{in} \quad W^{-1,s}(\Omega).$$

The proof of the Theorem is complete.

5. Appendix. A local energy equality for weak solutions of linear parabolic equations with unbounded coefficients

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Omega$, and let $0 < T < +\infty$. For $t \in]0,T]$, put $Q_t = \Omega\times]0,t[$. We consider the problem

$$\frac{\partial u}{\partial t} - \operatorname{div}\Big((a+b)\nabla u\Big) = 0 \quad \text{in} \quad Q_T, \quad u = 0 \quad \text{on} \quad \partial\Omega \times [0,T], \tag{A.1}$$

where $|a| \leq \text{Const}$ in Q_T, and b is a non-negative, possibly unbounded function. Our aim is to prove a local energy equality for weak solutions of (A.1). This equality can be motivated by multiplying the differential equation in (A.1) by $u\zeta$ ($\zeta \in C^1_c(\Omega)$) and integrating by parts over Ω.

Proposition A (local energy equality) *Let $a \in L^\infty(Q_T)$ and let b be a measurable function in Q_T such that*

$$b \geq 0 \quad a.\,e.\ in\ \ Q_T, \quad b^2 \in L^\infty(0,T;L^1(\Omega)), \quad \nabla(b^{1/2}) \in [L^2(Q_T)]^2. \tag{A.2}$$

Let $u \in L^\infty(Q_T) \cap C_w([0,T];L^2(\Omega)) \cap L^2(0,T;W^{1,2}_0(\Omega))$ verify

$$\int_{Q_T} b|\nabla u|^2 < +\infty, \quad \exists\, u' \in L^2(0,T;W^{-1,4/3}(\Omega)), \tag{A.3}$$

$$\left.\begin{aligned} &\int_0^t \langle u', v\rangle_{W^{-1,4/3},W^{1,4}_0} ds + \int_{Q_t} (a+b)\nabla u\cdot\nabla v dx = 0 \quad \forall\, t \in [0,T],\\ &\quad \forall\, v \in L^2(0,T;W^{1,4}_0(\Omega)). \end{aligned}\right\} \tag{A.4}$$

Then

$$\left.\begin{aligned} &\frac{1}{2}\int_\Omega u^2(x,t)\zeta(x)dx + \int_{Q_t}(a+b)(|\nabla u|^2\zeta + u\nabla\cdot\nabla\zeta) = \\ &= \frac{1}{2}\int_\Omega u^2(x,0)\zeta(x)dx \quad \forall\, t\in[0,T], \quad \forall\, \zeta \in C^1_c(\Omega). \end{aligned}\right\} \tag{A.5}$$

We emphasize that $v = u$ is not an admissible test function in (A.4).

From Proposition A we draw a conclusion which has been fundamental to pass to the limit $\varepsilon \to 0$ in the L^1-term on the right hand side of the functional relation in (3.9).

Corollary *Let k be a measurable function in Q_T such that*

$$k \geq 0 \quad a.\,e.\ in\ Q_T, \quad k \in L^\infty(0,T;L^1(\Omega)), \quad \delta\int_{Q_T}\frac{|\nabla k|^2}{(1+k)^{1+\delta}} < +\infty \quad (0<\delta<1). \tag{A.6}$$

Let $u \in L^\infty(Q_T) \cap C_w([0,T];L^2(\Omega)) \cap L^2(0,T;W^{1,2}_0(\Omega))$ verify

$$\int_{Q_T} k^{1/2}|\nabla u|^2 < +\infty, \quad \exists\, u' \in L^2(0,T;W^{-1,4/3}(\Omega)), \tag{A.7}$$

$$\left.\begin{aligned} &\int_0^t \langle u', v\rangle_{W^{-1,4/3},W^{1,4}_0} ds + \int_{Q_t} k^{1/2}\nabla u\cdot\nabla v dx = 0 \quad \forall\, t \in [0,T],\\ &\forall\, v \in L^2(0,T;W^{1,4}_0(\Omega)). \end{aligned}\right\} \tag{A.8}$$

Then

$$\left.\begin{aligned} &\frac{1}{2}\int_\Omega u^2(x,t)\zeta(x)dx + \int_{Q_t} k^{1/2}\Big(|\nabla u|^2\zeta + u\nabla u\cdot\nabla\zeta\Big) = \\ &= \frac{1}{2}\int_\Omega u^2(x,0)\zeta(x)dx \quad \forall\, t\in[0,T], \quad \forall\, \zeta \in C^1_c(\Omega). \end{aligned}\right\} \tag{A.9}$$

Proof of the corollary. Define

$$a = k^{1/2} - (1+k)^{1/2}, \quad b = (1+k)^{1/2}.$$

Then

$$a + b = k^{1/2}, \quad -1 \le a \le 0 \quad \text{a. e. in} \quad Q_T.$$

By (A.6) $\big($with $\delta = \frac{1}{2}$ therein$\big)$,

$$b \ge 1 \quad \text{a. e. in } Q_T, \quad b^2 \in L^\infty(0,T;L^1(\Omega)), \quad \int_{Q_T} |\nabla(b^{1/2})|^2 < +\infty.$$

Thus, (A.6) and (A.7) permit to apply Proposition A to (A.8). This gives (A.9).

We notice that the decomposition

$$k^{1/2} = \Big(k^{1/2} - (1+k)^{1/2}\Big) + (1+k)^{1/2}, \quad \text{where} \quad \nabla(1+k)^{1/4} \in L^2,$$

has been used in [14], [15] for the study of the steady case of the model problem (1.12), (1.13) (cf. the paper in [7], where the coefficient (eddy viscosity) $k^{1/2}$ is not included). ■

Before passing to the proof of Proposition A, we introduce more notations (cf. Section 2.2). Let $w \in L^p(0,T;X)$ $(1 \le p < +\infty)$. Given any $t_0 \in]0,T[$, for $\lambda \in]0, T-t_0[$ we introduce the *Steklov mean* w_λ *of* w

$$w_\lambda(t) = \frac{1}{\lambda}\int_t^{t+\lambda} w(s)ds, \quad t \in [0,t_0].$$

The following properties of w_λ are well-known.

(i) $w_\lambda \to w$ in $L^p(0,T;X)$ as $\lambda \to 0$;

(ii) there exists the distributional derivative $w'_\lambda \in L^p(0,t_0;X)$, where

$$w'_\lambda(t) = \frac{1}{\lambda}(w(t+\lambda) - w(t)) \quad \text{for a. e.} \quad t \in [0,t_0]$$

(cf., e. g., [3; Appendice], [5]).

Let H be a real Hilbert space with scalar product $(\cdot,\cdot)_H$ and continuous embedding $X \subset H$. Let $w \in L^p(0,T;X)$ $(2 \le p < +\infty)$ has the distributional derivative $w' \in L^{p'}(0,T;X^*)$. Then, for every $v \in L^p(0,T;X)$,

$$\int_0^{t_0} \langle w', v\rangle_{X^*,X} dt = \lim_{\lambda\to 0}\frac{1}{\lambda}\int_0^{t_0}\Big(w(t+\lambda) - w(t), v_\lambda(t)\Big)_H dt. \tag{A.10}$$

■

Proof of Proposition A Let $\omega_\rho(x) = \dfrac{1}{\rho^2}\omega\Big(\dfrac{x}{\rho}\Big)$ $(x \in \mathbb{R}^2, \rho > 0)$ denote the standard mollifying kernel. We extend $u(\cdot,t)$ by zero onto $\mathbb{R}^2 \setminus \Omega$ and denote this extension again by $u(\cdot,t)$.

Let $\zeta \in C^1(\mathbb{R}^2)$, supp$(\zeta) \subset \Omega$. Put $d_\zeta = \text{dist}(\text{supp}(\zeta), \partial\Omega)$. For $(x,t) \in \mathbb{R}^2\times]0,T[$, define

$$(\omega_\rho * u)(x,t) := (\omega_\rho * u(\cdot,t))(x) = \int_{\mathbb{R}^2}\omega_\rho(x-y)u(y,t)dy,$$

$$U_\rho(x,t) = \zeta(x)(\omega_\rho * u)(x,t).$$

Then for every $0 < \rho < \frac{1}{2}d_\zeta$ the function $v = \omega_\rho * U_\rho$ is in $L^2(0,T;W_0^{1,4}(\Omega))$. Hence, by (A.10).

$$\int_0^{t_0} \langle u', \omega_\rho * U_\rho\rangle_{W^{-1,4/3},W_0^{1,4}} dt =$$
$$= \lim_{\lambda\to 0} \frac{1}{\lambda}\int_0^{t_0}\int_\Omega [u(x,t+\lambda) - u(x,t)](\omega_\rho * U_\rho)_\lambda(x,t)dxdt \quad (t_0 \in]0,T[). \tag{A.11}$$

Let $\lambda \in]0, T - t_0[$. By Fubini's theorem, for all $(x,t) \in \Omega\times]0,t_0[$,

$$(\omega_\rho * U_\rho)_\lambda(x,t) = \Big[\omega_\rho * (U_\rho)_\lambda(\cdot,t)\Big](x), \quad (U_\rho)_\lambda(x,t) = \zeta(x)(\omega_\rho * u_\lambda(\cdot,t))(x).$$

It follows

$$\frac{1}{\lambda}\int_\Omega \Big[u(x,t+\lambda) - u(x,t)\Big](\omega_\rho * U_\rho)_\lambda(x,t)dx =$$
$$= \frac{1}{\lambda}\int_\Omega \Big[(\omega_\rho * u(\cdot,t+\lambda))(x) - (\omega_\rho * u(\cdot,t))(x)\Big](U_\rho)_\lambda(x,t)dx$$
$\Big($by Fubini's theorem; notice that $\operatorname{supp}(\zeta)\cap B_\rho(x) = \emptyset \quad \forall\, x\in\Omega,\ \operatorname{dist}(x,\partial\Omega) < \frac{1}{2}d_\zeta\Big)$
$$= \int_\Omega \Big\{\frac{\partial}{\partial t}(\omega_\rho * u_\lambda(\cdot,t))(x)\Big\}\zeta(x)(\omega_\rho * u_\lambda(\cdot,t))(x)$$
$$= \frac{1}{2}\frac{d}{dt}\int_\Omega [(\omega_\rho * u_\lambda(\cdot,t))(x)]^2\zeta(x)dx$$

and therefore

$$\frac{1}{\lambda}\int_0^{t_0}\int_\Omega [u(x,t+\lambda) - u(x,t)](\omega_\rho * U_\rho)_\lambda(x,t)dxdt =$$
$$= \frac{1}{2}\int_\Omega \Big[(\omega_\rho * u_\lambda(\cdot,t_0))(x)\Big]^2\zeta(x)dx - \frac{1}{2}\int_\Omega\Big[(\omega_\rho * u_\lambda(\cdot,0))(x)\Big]^2\zeta(x)dx. \tag{A.12}$$

Let be $0 < \rho < \frac{1}{2}d_\zeta$ and $0 \le t \le t_0$ (fixed). We prove

$$\lim_{\lambda\to 0}\int_\Omega [(\omega_\rho * u_\lambda(\cdot,t))(x)]^2\zeta(x)dx = \int_\Omega [(\omega_\rho * u(\cdot,t))(x)]^2\zeta(x)dx. \tag{A.13}$$

To see this, we firstly show

$$\lim_{\lambda\to 0}(\omega_\rho * u_\lambda(\cdot,t))(x) = (\omega_\rho * u(\cdot,t))(x) \quad \forall\, x \in \operatorname{supp}(\zeta).$$

Observing that $u \in C_w([0,T];L^2(\Omega))$, for every $\varepsilon > 0$ we find $\delta_{x,\varepsilon} > 0$ such that

$$\Big|\int_\Omega \omega_\rho(x-y)u(y,s)dy - \int_\Omega \omega_\rho(x-y)u(y,t)dy\Big| \le \varepsilon \quad \forall\, s\in[0,t_0],\ |s-t| \le \delta_{x,\varepsilon}.$$

Thus,

$$\Big|(\omega_\rho * u_\lambda(\cdot,t))(x) - (\omega_\rho * u(\cdot,t))(x)\Big| =$$
$$= \frac{1}{\lambda}\Big|\int_t^{t+\lambda}\int_\Omega \Big[\omega_\rho(x-y)u(y,s) - \omega_\rho(x-y)u(y,t)\Big]dyds\Big| \le \varepsilon \quad \forall\, 0 < \lambda < \min\Big\{\frac{1}{2}d_\zeta, \delta_{x,\varepsilon}\Big\}.$$

Secondly, we insert $v = \omega_\rho * U_\rho$ into (A.4) and use (A.11), combined with (A.12), (A.13). This gives, for all $t \in [0,T]$,

$$\frac{1}{2}\int_\Omega [(\omega_\rho * u(\cdot,t))(x)]^2\zeta(x)dx + \int_{Q_t}(a+b)\nabla u\cdot\nabla(\omega_\rho * U_\rho) = \frac{1}{2}\int_\Omega [(\omega_\rho * u(\cdot,0))(x)]^2\zeta(x)dx \tag{A.14}$$

$\Big($notice that $\int_\Omega [(\omega_\rho * u(\cdot, t_0))(x)]^2 \zeta(x)dx \to \int_\Omega [(\omega_\rho * u(\cdot, T))(x)]^2 \zeta(x)dx$ as $t_0 \to T\Big)$.
To carry out the passage to the limit $\rho \to 0$ in (A.14) we observe that $u(\cdot, t) \in L^2(\Omega)$ $(t \in [0, T])$ and $u \in L^2(Q_T)$ imply

$$\omega_\rho * u(\cdot, t) \to u(\cdot, t) \quad \text{strongly in} \quad L^2(\Omega), \quad \omega_\rho * U_\rho \to \zeta u \quad \text{strongly in} \quad L^2(Q_T) \quad \text{as} \quad \rho \to 0.$$

Then we show

$$\|\nabla(\omega_\rho * U_\rho)\|_{[L^2(Q_T)]^2} \le c, \quad \|b^{1/2}\nabla(\omega_\rho * U_\rho)\|_{[L^2(Q_T)]^2} \le c,$$

where the constants c do not depend on $0 < \rho < \frac{1}{2}d_\zeta$. This can be proved by following the idea in [7; pp. 1060-1061]. Then (A.5) is readily obtained from (A.14) by $\rho \to 0$.

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