Boll. Accademia Gioenia di Scienze Naturali - Catania

BOLLAG Vol. 46, N. 376 (2013), pp. 44 - 56
ISSN 0393-7143

Anno di fondazione 1824

Optimization problems for eigenvalues of p-Laplace equations with indefinite weights [†]

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Dedicated to Prof. Mario Marino on the occasion of his 70-th birthday.

Summary

This paper is concerned with minimization and maximization of the principal eigenvalue of p-Laplace equations depending on functions which belong to a class of rearrangements. In case of $p = 2$, this optimization problems are motivated by the question of determining the most convenient spatial arrangement of favorable and unfavorable resources for species to survive or to decline. We prove existence and uniqueness results, and present some features of optimizers. The radial case is discussed in detail.

Key words: *p-Laplace equations, Principal eigenvalue, Rearrangements, Optimization*

Riassunto

Si studia il minimo ed il massimo del primo autovalore di un'equazione col p-Laplaciano contenente un peso variabile in una classe di riordinamenti. Nel caso $p = 2$, questi problemi di ottimizzazione sono motivati dalla ricerca della distribuzione piú conveniente delle risorse in un determinato ambiente affinché si abbia la sopravvivenza o la estinzione di una specie. Si trovano risultati di esistenza, unicitá, e rappresentazione degli estremanti. Si studia, in particolare, il caso di domini radiali.

Parole chiave: *p-Laplaciano, Primo autovalore, Riordinamenti, Ottimizzazione*

1 Introduction

Suppose that $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain representing a region occupied by a population that diffuses at rate D and grows or declines locally at a rate $g(x)$, so that $g(x) > 0$ corresponds to local growth and $g(x) < 0$ to local decline. Suppose that the exterior of Ω is hostile to the population (individuals which across the boundary die). Suppose that the carrying capacity of the

[†]Lecture given on the occasion of the 70th birthday of Mario Marino, 3-4 May 2013, Catania

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population is equal to K. If $\varphi(x,t)$ is the population density, the global behavior of the population is described by the diffusion equation

$$\frac{\partial\varphi}{\partial t} = D\Delta\varphi + g(x)\varphi - K\varphi^2 \quad \text{in} \quad \Omega\times(0,T), \quad \varphi = 0 \quad \text{on} \quad \partial\Omega\times(0,T), \tag{1}$$

where $\Delta\varphi$ denotes the spatial Laplacian of $\varphi(x,t)$. A simplified form of the logistic equation (1) has been introduced by Pierre François Verhulst about 175 years ago.

As proved in [1], equation (1) predicts persistence of the population if and only if $\lambda_g < 1/D$, where λ_g is the (positive) principal eigenvalue in

$$\Delta u + \lambda g(x)u = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega. \tag{2}$$

The existence and variational characterization of the eigenvalues of (2) were established in [2]. Since the principal eigenvalue λ_g depends on g, it is very important to find its extreme values for weights within the set of rearrangements of a given weight function $g_0(x)$. This investigation has been done in the recent paper [3]. Related results were obtained in [4] and in [5]. Eigenvalues for equations with sign changing weights have been discussed in [6].

In the present paper we investigate a more general equation. Namely, let Ω be a bounded smooth domain in $\mathbb{R}^N$, and let $g \in L^\infty(\Omega)$ be a function (possibly sign changing) positive in a set of positive measure. For $p > 1$, we consider the eigenvalue problem

$$-\Delta_p u = \lambda\, g u^{p-1}, \quad u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega. \tag{3}$$

Here λ is the principal eigenvalue which depends on Ω, p and g. In what follows, Ω and p will be fixed, whereas, the function g may change, therefore we shall write $\lambda = \lambda_g$. It is well known that

$$\lambda_g = \inf_v \left\{ \frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega g|v|^p dx} : \ v \in H_0^{1,p}(\Omega), \ \int_\Omega g|v|^p dx > 0 \right\} = \frac{\int_\Omega |\nabla u_g|^p dx}{\int_\Omega g u_g^p dx},$$

where $u_g \in H_0^{1,p}(\Omega)$ is the principal (positive) eigenfunction, which we normalize so that

$$\int_\Omega u_g^p dx = 1.$$

It is known that the normalized eigenfunction u_g is positive and unique. For a discussion of existence and uniqueness of the principal eigenvalue of problem (3), we refer to [7].

If $E \subset \mathbb{R}^N$ is a measurable set we denote with $|E|$ its Lebesgue measure. We say that two measurable functions $f(x)$ and $g(x)$ defined in Ω have the same rearrangement if

$$|\{x \in \Omega : f(x) \geq \beta\}| = |\{x \in \Omega : g(x) \geq \beta\}| \ \ \forall \beta \in \mathbb{R}.$$

If $g_0 \in L^\infty(\Omega)$, we denote by $\mathcal{G} = \mathcal{G}(g_0)$ the class of its rearrangements. We assume $g_0(x) > 0$ in a subset of positive measure, and suppose g_0 is not a constant. Let $\overline{\mathcal{G}}$ be the closure of $\mathcal{G}$ in the weak* topology of $L^\infty(\Omega)$. Note that, even if $g_0(x) > 0$ in a subset of positive measure, we may have $g \in \overline{\mathcal{G}}$ with $g(x) \leq 0$ in Ω. In this case, the set of functions v such that $\int_\Omega g|v|^p dx > 0$ is empty, and we put $\lambda_g = +\infty$.

The paper is organized as follows. In Section 2, we investigate minimization and maximization of the principal eigenvalue λ_g for $g \in \mathcal{G}$. We also give a representation of minimizer and maximizer. In Section 3 we consider the radial case and find more precise results. Note that, in case of $p = 2$, we find all results from [3], however, the present approach is slightly simpler. In Section 4 we give an interpretation of our results for a population which diffuses according to equation (1).

2 Optimization of the principal eigenvalue

Let $\mathcal{G}$ be the class of rearrangements generated by a function $g_0(x) \in L^\infty(\Omega)$ which is positive in a subset of positive measure, and let $\overline{\mathcal{G}}$ be the closure of $\mathcal{G}$ in the weak* topology of $L^\infty(\Omega)$. For $g \in \overline{\mathcal{G}}$ with $g(x) > 0$ in a subset of positive measure, we consider the problem

$$\inf_{g\in\mathcal{G}} \lambda_g = \inf_{g\in\mathcal{G}} \frac{\int_\Omega |\nabla u_g|^p dx}{\int_\Omega g u_g^p dx}, \tag{4}$$

where u_g is a positive eigenfunction of problem (3) corresponding to g. Note that

$$\inf_{g\in\mathcal{G}} \lambda_g = \inf_{g\in\mathcal{G}} \inf_v \left\{ \frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega g|v|^p dx} : \ v \in H_0^{1,p}(\Omega), \ \int_\Omega g|v|^p dx > 0 \right\}. \tag{5}$$

In case of $g \in \overline{\mathcal{G}}$ with $g(x) \le 0$ in Ω we put $\lambda_g = +\infty$. For $g \in \overline{\mathcal{G}}$, we define

$$J(g) = \frac{1}{\lambda_g}.$$

Of course, when $g(x) \le 0$ in Ω we have $J(g) = 0$. Otherwise, we have

$$J(g) = \sup_{v\in H_0^{1,p}(\Omega)} \frac{\int_\Omega g|v|^p dx}{\int_\Omega |\nabla v|^p dx} = \frac{\int_\Omega g u_g^p dx}{\int_\Omega |\nabla u_g|^p dx}. \tag{6}$$

Note that problem (4) is equivalent to problem

$$\sup_{g\in\mathcal{G}} J(g) = \sup_{g\in\mathcal{G}} \sup_{v\in H_0^{1,p}(\Omega)} \frac{\int_\Omega g|v|^p dx}{\int_\Omega |\nabla v|^p dx}. \tag{7}$$

We also investigate the problem

$$\sup_{g\in\mathcal{G}} \lambda_g = \sup_{g\in\mathcal{G}} \frac{\int_\Omega |\nabla u_g|^p dx}{\int_\Omega g u_g^p dx}. \tag{8}$$

Note that, problem (8) is equivalent to problem

$$\inf_{g\in\mathcal{G}} J(g) = \inf_{g\in\mathcal{G}} \sup_{v\in H_0^{1,p}(\Omega)} \frac{\int_\Omega g|v|^p dx}{\int_\Omega |\nabla v|^p dx}. \tag{9}$$

We will see that problems (7) and (9) are quite different. In our discussion, we make use of the following strong results proved in [8] and [9]. For short, throughout the paper we shall write increasing instead of non-decreasing, and decreasing instead of non-increasing.

Lemma 1 *Let $g : \Omega \to \mathbb{R}$ and $w : \Omega \to \mathbb{R}$ be measurable functions, and suppose that every level set of w has measure zero. Then there exists an increasing function ϕ such that $\phi(w)$ is a rearrangement of g. Furthermore, there exists a decreasing function ψ such that $\psi(w)$ is a rearrangement of g.*

Proof. The first assertion follows from Lemma 2.9 of [9]. The second assertion follows applying the first one to $-w$. □

Denote with $\overline{\mathcal{G}}$ the weak closure of $\mathcal{G}$ in $L^p(\Omega)$, $1 \le p$. It is well known that $\overline{\mathcal{G}}$ is convex and weakly sequentially compact (see for example Lemma 2.2 of [9]).

Lemma 2 *Let $\mathcal{G}$ be the set of rearrangements of a fixed function $g_0 \in L^p(\Omega)$, $p \geq 1$, and let $w \in L^q(\Omega)$, $q = p/(p-1)$. If there is an increasing function ϕ such that $\phi(w) \in \mathcal{G}$ then*

$$\int_\Omega g\, w\, dx \leq \int_\Omega \phi(w)\, w\, dx \quad \forall g \in \overline{\mathcal{G}},$$

and the function $\phi(w)$ is the unique maximizer relative to $\overline{\mathcal{G}}$. Furthermore, if there is a decreasing function ψ such that $\psi(w) \in \mathcal{G}$ then

$$\int_\Omega g\, w\, dx \geq \int_\Omega \psi(w)\, w\, dx \quad \forall g \in \overline{\mathcal{G}},$$

and the function $\psi(w)$ is the unique minimizer relative to $\overline{\mathcal{G}}$.

Proof. The first assertion follows from Lemma 2.4 of [9]. The the second assertion follows from the first one putting $\phi(t) = \psi(-t)$. □

Lemma 3 *Let $\mathcal{G}$ denote the set of rearrangements of a fixed function $g_0 \in L^p(\Omega)$, $p \geq 1$. Let $\Psi : L^p(\Omega) \to \mathbb{R}$ be a convex functional sequentially continuous in the $L^q(\Omega)$ topology on $L^p(\Omega)$, $q = \frac{p}{p-1}$. Then Ψ attains a maximum value relative to $\mathcal{G}$.*

Proof. See Theorem 7 of [8]. □

We recall that the $L^q(\Omega)$ topology on $L^p(\Omega)$ is the weak topology if $1 \leq p < \infty$, and the weak* topology if $p = \infty$ [8].

Now, we prove some results about the map $g \mapsto J(g)$, where $J(g)$ is defined as in (6).

Lemma 4 *The map $g \mapsto J(g)$ is continuous with respect to the weak* topology in $L^\infty(\Omega)$.*

Proof. If $g(x) \leq 0$ in Ω then, $J(g) = 0$. We claim that, if $g_i \rightharpoonup g$ in the weak* topology of $L^\infty(\Omega)$ then $J(g_i) \to 0$ as $i \to \infty$. By contradiction, suppose there is a subsequence (denoted again g_i) such that $J(g_i) \geq \epsilon$ for some positive ϵ. For each i, we must have $g_i > 0$ in a set of positive measure. From

$$\frac{\int_\Omega g_i u_{g_i}^p\, dx}{\int_\Omega |\nabla u_{g_i}|^p dx} \geq \epsilon, \tag{10}$$

recalling the normalization of u_{g_i} and the fact that $g_i(x) \leq M$ for some constant M we get,

$$\int_\Omega |\nabla u_{g_i}|^p dx \leq \frac{1}{\epsilon} \int_\Omega g_i u_{g_i}^p\, dx \leq \frac{M}{\epsilon}.$$

It follows that, for a suitable subsequence (denoted again u_{g_i}), $u_{g_i} \rightharpoonup \overline{u}$ weakly in the $H^{1,p}(\Omega)$ topology and $u_{g_i} \to \overline{u}$ in the $L^s(\Omega)$ norm for some $s > p$, and the $L^p(\Omega)$ norm of $\overline{u}$ is one. We have

$$\liminf_{i\to\infty} \int_\Omega |\nabla u_{g_i}|^p dx \geq \int_\Omega |\nabla \overline{u}|^p dx > 0, \tag{11}$$

and

$$\lim_{i\to\infty} \int_\Omega g_i u_{g_i}^p\, dx = \int_\Omega g \overline{u}^p\, dx.$$

Since $\int_\Omega g_i u_{g_i}^p\, dx > 0$, and $\int_\Omega g \overline{u}^p\, dx \leq 0$, we must have

$$\lim_{i\to\infty} \int_\Omega g_i u_{g_i}^p\, dx = 0.$$

The last result and (11) contradict (10), hence, $J(g_i) \to 0 = J(g)$.

Now, let $g(x) > 0$ in a set of positive measure, and let $g_i \rightharpoonup g$ in the weak* topology in $L^\infty(\Omega)$. We may assume that $g_i > 0$ in a set of positive measure (depending on i). If u_{g_i}, u_g are normalized as usual, we have

$$J(g_i) = \frac{\int_\Omega g_i u_{g_i}^p \, dx}{\int_\Omega |\nabla u_{g_i}|^p dx} \geq \frac{\int_\Omega g_i u_g^p \, dx}{\int_\Omega |\nabla u_g|^p dx} = J(g) \frac{\int_\Omega g_i u_g^p \, dx}{\int_\Omega g u_g^p \, dx}. \tag{12}$$

Let $0 < \epsilon < J(g)$. Since

$$\lim_{i\to\infty} \int_\Omega g_i u_g^p \, dx = \int_\Omega g u_g^p \, dx,$$

by (12) we have

$$J(g_i) > J(g) - \epsilon \ \text{ for } \ i > \nu_\epsilon. \tag{13}$$

As a consequence,

$$\int_\Omega |\nabla u_{g_i}|^p dx \leq \frac{1}{J(g) - \epsilon} \int_\Omega g_i u_{g_i}^p \, dx \leq C. \tag{14}$$

It follows that, for a suitable subsequence (denoted again u_{g_i}), $u_{g_i} \rightharpoonup \overline{u}$ weakly in the $H^{1,p}(\Omega)$ topology and $u_{g_i} \to \overline{u}$ in the $L^s(\Omega)$ norm for some $s > p$. Hence,

$$\liminf_{i\to\infty} \int_\Omega |\nabla u_{g_i}|^p dx \geq \int_\Omega |\nabla \overline{u}|^p dx,$$

and

$$\lim_{i\to\infty} \int_\Omega g_i u_{g_i}^p \, dx = \int_\Omega g \overline{u}^p \, dx.$$

Due to our normalization, the $L^p(\Omega)$ norm of $\overline{u}$ is one. Using the last two results we find

$$\limsup_{i\to\infty} J(g_i) = \limsup_{i\to\infty} \frac{\int_\Omega g_i u_{g_i}^p \, dx}{\int_\Omega |\nabla u_{g_i}|^p dx} \leq \frac{\int_\Omega g \overline{u}^p \, dx}{\int_\Omega |\nabla \overline{u}|^p dx} \leq J(g). \tag{15}$$

From (13) and (15) it follows that $J(g_i) \to J(g)$. We also remark that our proof yields $u_{g_i} \to u_g$ in the norm of $H^{1,p}(\Omega)$. The proof of the lemma is complete. □

Lemma 5 *The map $g \mapsto J(g)$ is Gateaux differentiable.*

Proof. If $g, \ g_i \in \overline{\mathcal{G}}$ we have

$$\begin{aligned} J(g) + \frac{\int_\Omega (g_i - g) u_g^p \, dx}{\int_\Omega |\nabla u_g|^p dx} &= \frac{\int_\Omega g_i u_g^p \, dx}{\int_\Omega |\nabla u_g|^p dx} \leq J(g_i) = \frac{\int_\Omega g_i u_{g_i}^p \, dx}{\int_\Omega |\nabla u_{g_i}|^p dx} \\ = \frac{\int_\Omega g u_{g_i}^p \, dx}{\int_\Omega |\nabla u_{g_i}|^p dx} + \frac{\int_\Omega (g_i - g) u_{g_i}^p \, dx}{\int_\Omega |\nabla u_{g_i}|^p dx} &\leq J(g) + \frac{\int_\Omega (g_i - g) u_{g_i}^p \, dx}{\int_\Omega |\nabla u_{g_i}|^p dx}. \end{aligned} \tag{16}$$

Let $t_i > 0$ be a sequence such that $t_i \to 0$ as $i \to \infty$. Let $g, h \in \overline{\mathcal{G}}$ and let $g_i = g + t_i(h - g)$. Then, by (16) we find

$$\begin{aligned} J(g) + t_i \frac{\int_\Omega (h - g) u_g^p \, dx}{\int_\Omega |\nabla u_g|^p dx} &\leq J(g_i) \\ \leq J(g) + t_i \frac{\int_\Omega (h - g) u_{g_i}^p \, dx}{\int_\Omega |\nabla u_{g_i}|^p dx}&. \end{aligned} \tag{17}$$

Recall that we are using the normalization $\int_\Omega u_{g_i}^p\,dx = 1$. Since $t_i \to 0$ as $i \to \infty$, we have $g_i \to g$ in the norm of $L^\infty(\Omega)$. As a consequence, by the proof of Lemma 4, the sequence u_{g_i} converges, in the norm of $H^{1,p}(\Omega)$, to u_g. Therefore, from (17) we get

$$\lim_{t\to 0^+} \frac{J(g+t(h-g))-J(g)}{t} = \frac{\int_\Omega (h-g)u_g^p\,dx}{\int_\Omega |\nabla u_g|^p dx}. \tag{18}$$

The lemma follows. □

Lemma 6 *The functional $J(g)$ is convex; furthermore, if $\int_\Omega g_0(x)dx \geq 0$ then $J(g)$ is strictly convex.*

Proof. Let $f, g \in \overline{\mathcal{G}}$, let $0 < t < 1$ and let $w \in H_0^{1,p}(\Omega)$. We have

$$\frac{\int_\Omega (tf+(1-t)g)|w|^p\,dx}{\int_\Omega |\nabla w|^p dx} = t\frac{\int_\Omega f|w|^p\,dx}{\int_\Omega |\nabla w|^p dx} + (1-t)\frac{\int_\Omega g|w|^p\,dx}{\int_\Omega |\nabla w|^p dx}.$$

By taking the superior of both sides relative to $w \in H_0^{1,p}(\Omega)$, we get

$$J(tf+(1-t)g)) \leq tJ(f) + (1-t)J(g),$$

that is, the convexity.

Now, suppose $\int_\Omega g_0(x)dx \geq 0$. Then, $\int_\Omega g(x)dx \geq 0$ for all $g \in \overline{\mathcal{G}}$. For $f, g \in \overline{\mathcal{G}}$, assume equality holds in the above inequality for some $t \in (0,1)$. Then,

$$\frac{\int_\Omega (tf+(1-t)g)u_t^p\,dx}{\int_\Omega |\nabla u_t|^p dx} = t\frac{\int_\Omega f u_f^p\,dx}{\int_\Omega |\nabla u_f|^p dx} + (1-t)\frac{\int_\Omega g u_g^p\,dx}{\int_\Omega |\nabla u_g|^p dx}.$$

where u_t, u_f and u_g are normalized functions corresponding to $J(tf+(1-t)g)$, $J(f)$ and $J(g)$ respectively. Since

$$\frac{\int_\Omega (tf+(1-t)g)u_t^p\,dx}{\int_\Omega |\nabla u_t|^p dx} = t\frac{\int_\Omega f u_t^p\,dx}{\int_\Omega |\nabla u_t|^p dx} + (1-t)\frac{\int_\Omega g u_t^p\,dx}{\int_\Omega |\nabla u_t|^p dx},$$

it follows that

$$\frac{\int_\Omega f u_f^p\,dx}{\int_\Omega |\nabla u_f|^p dx} = \frac{\int_\Omega f u_t^p\,dx}{\int_\Omega |\nabla u_t|^p dx}, \quad \frac{\int_\Omega g u_g^p\,dx}{\int_\Omega |\nabla u_g|^p dx} = \frac{\int_\Omega g u_t^p\,dx}{\int_\Omega |\nabla u_t|^p dx}.$$

By the uniqueness of the normalized eigenfunction corresponding to $J(f)$ and to $J(g)$, we must have $u_t = u_f = u_g$.

Consider first the case f and g are positive in a set of positive measure. Then, $J(f) > 0$ and $J(g) > 0$. Since

$$-\Delta_p u_f = \frac{1}{\sqrt{J(f)}} f u_f^{p-1} \ \text{ a.e. in } \ \Omega,$$

and

$$-\Delta_p u_g = \frac{1}{\sqrt{J(g)}} g u_g^{p-1} \ \text{ a.e. in } \ \Omega,$$

with $u_f = u_g$, we have

$$\frac{1}{\sqrt{J(f)}} f(x) = \frac{1}{\sqrt{J(g)}} g(x) \ \text{ a.e. in } \ \Omega.$$

Integrating over Ω we find $J(f) = J(g)$. Hence, $f(x) = g(x)$ a.e. in Ω.

Consider now the case $g \leq 0$. Since $\int_\Omega g(x)dx \geq 0$, we must have $g = 0$. Then, $u_g = 0$, and the previous proof shows that also $u_f = 0$. But, $u_f = 0$ implies $f \leq 0$. Finally, since $\int_\Omega f(x)dx \geq 0$, we must have $f = 0 = g$. The proof of the lemma is complete. □

Now we prove the main result of this section.

Theorem 1 *Let g_0 be a bounded function in Ω, positive in a subset of positive measure. Let $\mathcal{G}$ be the class of rearrangements generated by g_0, and let $J(g)$ be defined as in* (6).
(i) The problem

$$\max_{g\in\mathcal{G}} J(g)$$

has a solution; furthermore, if $\hat{g} \in \mathcal{G}$ is a maximizer then $\hat{g} = \phi(u_{\hat{g}})$ for some increasing function $\phi(t)$.
(ii) The problem

$$\min_{g\in\overline{\mathcal{G}}} J(g)$$

has a solution; if $\int_\Omega g_0(x)dx \geq 0$, the minimizer $\check{g}$ is unique; if $\int_\Omega g_0(x)dx > 0$, we have $\check{g} = \psi(u_{\check{g}})$ for some decreasing function $\psi(t)$; finally, if $g_0(x) \geq 0$ then the minimizer $\check{g}$ belongs to $\mathcal{G}$.

Proof. Since $J(g)$ is continuous with respect to the weak* topology of $L^\infty(\Omega)$ (by Lemma 4) and since it is convex (by Lemma 6), a maximizer $\hat{g}$ of $J(g)$ exists on $\mathcal{G}$ (by Lemma 3). Since $J(g)$ is Gâteaux differentiable (by Lemma 5), if $0 < t < 1$ and if $g_t = \hat{g} + t(g - \hat{g})$, we have

$$J(\hat{g}) \geq J(g_t) = J(\hat{g}) + t\frac{\int_\Omega (g-\hat{g})u_{\hat{g}}^p dx}{\int_\Omega |\nabla u_{\hat{g}}|^p dx} + o(t) \quad \text{as} \quad t \to 0.$$

It follows that

$$\frac{\int_\Omega (g-\hat{g})u_{\hat{g}}^p dx}{\int_\Omega |\nabla u_{\hat{g}}|^p dx} + \frac{o(t)}{t} \leq 0.$$

As $t \to 0$, we find

$$\int_\Omega (g-\hat{g})u_{\hat{g}}^p dx \leq 0.$$

Equivalently, we have

$$\int_\Omega g\, u_{\hat{g}}^p dx \leq \int_\Omega \hat{g}\, u_{\hat{g}}^p dx \quad \forall g \in \overline{\mathcal{G}}. \tag{19}$$

Recall that $u_{\hat{g}}(x) > 0$ a.e. in Ω, because $\hat{g} \in \mathcal{G}$ and $\mathcal{G}$ is generated by a function g_0 which is positive in a subset of positive measure. Hence, $J(\hat{g}) > 0$ and $u_{\hat{g}}$ satisfies the equation

$$-\Delta_p u_{\hat{g}} = \frac{1}{\sqrt{J(\hat{g})}} \hat{g}\, u_{\hat{g}}^{p-1}. \tag{20}$$

By equation (20), the function $u_{\hat{g}}$ cannot have flat zones neither in the set $F_1 = \{x \in \Omega : \hat{g}(x) < 0\}$ nor in the set $F_2 = \{x \in \Omega : \hat{g}(x) > 0\}$. By Lemma 1, there is an increasing function $\phi_1(t)$ such that $\phi_1(u_{\hat{g}}^p)$ is a rearrangement of $\hat{g}(x)$ on $F_1 \cup F_2$. Define

$$\alpha = \inf_{x\in\Omega\setminus F_1} u_{\hat{g}}^p(x).$$

Using (19), one proves that $u_{\hat{g}}^p(x) \leq \alpha$ in F_1 (see [10] for details). Define

$$\beta = \sup_{x\in\Omega\setminus F_2} u_{\hat{g}}^p(x).$$

Using (19) again one shows that $u_{\hat{g}}^p(x) \geq \beta$ in F_2. Now we put

$$\tilde{\phi}(t) = \begin{cases} \phi_1(t) & \text{if } 0 \leq t < \alpha \\ 0 & \text{if } \alpha \leq t \leq \beta \\ \phi_1(t) & \text{if } t > \beta. \end{cases}$$

The function $\tilde\phi(t)$ is increasing and $\tilde\phi(u^p_{\hat g})$ is a rearrangement of $\hat g(x)$ in Ω. Indeed, the functions $\hat g$ and $\tilde\phi(u^p_{\hat g})$ have the same rearrangement on $F_1\cup F_2$, and both vanish on $\Omega\setminus(F_1\cup F_2)$. By (19) and Lemma 2 we must have $\hat g=\tilde\phi(u^p_{\hat g})$. Part (i) of the theorem follows with $\phi(t)=\tilde\phi(t^p)$.

Since the functional $J(g)$ is continuous with respect to the weak* topology of $L^\infty(\Omega)$, and since $\overline{\mathcal G}$ is weakly compact, a minimizer $\breve g$ exists in $\overline{\mathcal G}$. Assuming $\int_\Omega g_0(x)dx\ge 0$, the uniqueness of the minimizer follows from the strict convexity of $J(g)$ (see Lemma 6). If $\int_\Omega g_0(x)dx>0$, the minimizer $\breve g$ is positive in a subset of positive measure. Therefore, $J(\breve g)>0$ and $u_{\breve g}(x)>0$ a.e. in Ω. If $0<t<1$ and if $g_t=\breve g+t(g-\breve g)$, since $J(g)$ is differentiable, we have

$$J(\breve g)\le J(g_t)=J(\breve g)+t\frac{\int_\Omega(g-\breve g)u^p_{\breve g}dx}{\int_\Omega|\nabla u_{\breve g}|^pdx}+o(t)\quad\text{as}\ \ t\to 0.$$

It follows that

$$\int_\Omega(g-\breve g)u^p_{\breve g}dx\ge 0.$$

Equivalently, we have

$$\int_\Omega g\, u^p_{\breve g}dx\ge\int_\Omega\breve g\, u^p_{\breve g}dx\quad\forall g\in\overline{\mathcal G}. \tag{21}$$

The function $u_{\breve g}$ satisfies the equation

$$-\Delta_p u_{\breve g}=\frac{1}{\sqrt{J(\breve g)}}\breve g\, u^{p-1}_{\breve g}. \tag{22}$$

By equation (22), the function $u_{\breve g}$ cannot have flat zones neither in the set $F_3=\{x\in\Omega:\breve g(x)>0\}$ nor in the set $F_4=\{x\in\Omega:\breve g(x)<0\}$. By Lemma 1, there is a decreasing function $\psi_1(t)$ such that $\psi_1(u^p_{\breve g})$ is a rearrangement of $\breve g(x)$ on $F_3\cup F_4$. Following the proof of Theorem 2.1 of [10], we introduce the class $\mathcal W$ of rearrangements of our minimizer $\breve g$. Of course, $\mathcal W\subset\overline{\mathcal G}$. Define

$$\gamma=\inf_{x\in\Omega\setminus F_3}u^p_{\breve g}(x).$$

Using (21), one proves that $u^p_{\breve g}(x)\le\gamma$ in F_3. Define

$$\delta=\sup_{x\in\Omega\setminus F_4}u^p_{\breve g}(x).$$

Using (21) again one shows that $u^p_{\breve g}(x)\ge\delta$ in F_4. Now we put

$$\tilde\psi(t)=\begin{cases}\psi_1(t) & \text{if}\ \ 0\le t<\gamma\\ 0 & \text{if}\ \ \gamma\le t\le\delta\\ \psi_1(t) & \text{if}\ \ t>\delta.\end{cases}$$

The function $\tilde\psi(t)$ is decreasing and $\tilde\psi(u^p_{\breve g})$ is a rearrangement of $\breve g(x)$ in Ω. Indeed, the functions $\breve g$ and $\tilde\psi(u^p_{\breve g})$ have the same rearrangement on $F_3\cup F_4$, and both vanish on $\Omega\setminus(F_3\cup F_4)$. By (21) and Lemma 2 we must have $\breve g=\tilde\psi(u^p_{\breve g})\in\mathcal W$.

Note that, in general, the minimizer $\breve g$ does not belong to $\mathcal G$ (see next theorem). Assuming $g_0(x)\ge 0$, we can prove that $\breve g\in\mathcal G$. Indeed, by (22), the function $u_{\breve g}$ cannot have flat zones in the set $F=\{x\in\Omega:\breve g(x)>0\}$. If $|F|<|\Omega|$, since $\breve g\in\overline{\mathcal G}$, by Lemma 2.14 of [9] we have $|F|\ge|\{x\in\Omega:g_0(x)>0\}|$. Therefore there is $g_1\in\mathcal G$ such that its support is contained in F. By Lemma 1, there is a decreasing function $\psi_1(t)$ such that $\psi_1(u^p_{\breve g})$ is a rearrangement of $g_1(x)$ on F. Define

$$\gamma=\inf_{x\in\Omega\setminus F}u^p_{\breve g}(x).$$

Using (21), one proves that $u_{\breve{g}}^p(x) \leq \gamma$ in F. By using equation (21) once more we find that $u_{\breve{g}}^p(x) < \gamma$ a.e. in F. Now define

$$\tilde{\psi}(t) = \begin{cases} \psi_1(t) & \text{if } 0 \leq t < \gamma \\ 0 & \text{if } t \geq \gamma. \end{cases}$$

The function $\tilde{\psi}(t)$ is decreasing and $\tilde{\psi}(u_{\breve{g}}^p)$ is a rearrangement of $g_1 \in \mathcal{G}$ on Ω. Indeed, the functions g_1 and $\tilde{\psi}(u_{\breve{g}}^p)$ have the same rearrangement on F, and both vanish on $\Omega \setminus F$. By (21) and Lemma 2 we must have $\breve{g} = \tilde{\psi}(u_{\breve{g}}^p) \in \mathcal{G}$. Hence, in case of $|F| < |\Omega|$, the conclusion follows with $\psi(t) = \tilde{\psi}(t^p)$. If $|F| = |\Omega|$, the proof is easier and we do not need the introduction of the function g_1. The theorem follows. □

Remark. By the last assertion of the previous theorem, if $g_0(x) \geq 0$ then the minimizer $\breve{g}$ belongs to $\mathcal{G}$. We may ask what happens if $g_0(x)$ is sign changing. Well, again by the previous theorem, provided $\int_\Omega g_0(x)dx > 0$, we have $\breve{g} = \psi(u_{\breve{g}})$ for some decreasing function $\psi(t)$. In this situation, the function $\breve{g}$ (which belongs to the enlarged set $\overline{\mathcal{G}}$) cannot belong to $\mathcal{G}$ since $\breve{g}$ cannot be sign changing, as the following theorem shows.

Theorem 2 *Suppose $u \in H_0^{1,p}(\Omega) \cap H^2(\Omega) \cap C^0(\Omega)$ satisfies $u(x) > 0$ in Ω and*

$$-\Delta_p u = \Lambda\psi(u)u^{p-1} \quad \textit{a.e. in} \quad \Omega$$

for some positive Λ and some decreasing bounded function ψ. Then, either $\Delta u \leq 0$ or $\Delta u \geq 0$ a.e. in Ω.

Proof. By contradiction, suppose that the essential range of $\Delta_p u$ contains positive and negative values. Since $u > 0$ and $-\Delta_p u = \Lambda\psi(u)u^{p-1}$, $\psi(t)$ takes positive and negative values for $t > 0$. Let

$$\beta = \sup\{t : \psi(t) \geq 0\},$$

$$\Omega_\beta = \{x \in \Omega : u(x) > \beta\}.$$

By our assumptions, the open set Ω_β is not empty. On the other side, since ψ is decreasing and $u > 0$ we have

$$-\Delta_p u < 0 \;\text{ in }\; \Omega_\beta, \quad u = \beta \;\text{ on }\; \partial\Omega_\beta.$$

The maximum principle for p-subharmonic functions yields $u(x) \leq \beta$ in Ω_β. This contradicts the definition of Ω_β, and the theorem follows. □

3 The radial case

In this section, let $\Omega = B$ be a ball centered in the origin. A function u defined in B is Schwarz symmetric if and only if all sets $\{x \in \Omega : u(x) > t\}$, $\forall t \in \mathbb{R}$, are balls centered in the origin. In this case we write $u = u^\sharp$. If u is not Schwarz symmetric, we associate to u a rearrangement $u^\sharp$ which is Schwarz symmetric. The function $u^\sharp$ is named the Schwarz decreasing rearrangement of u. We also use the function $u_\sharp$, a rearrangement of u which is radially symmetric and increasing with respect to $|x|$. The following results are well known.

Lemma 7 *Let $\Omega = B$.*
i) If $f(x)$ and $g(x)$ belong to $L^\infty(\Omega)$ then

$$\int_\Omega f^\sharp(x)g_\sharp(x)dx \leq \int_\Omega f(x)g(x)dx \leq \int_\Omega f^\sharp(x)g^\sharp(x)dx. \tag{23}$$

ii) If $u \in H_0^{1,p}(\Omega)$, $u(x) \geq 0$ *then* $u^\sharp \in H_0^{1,p}(\Omega)$, $u^\sharp(x) \geq 0$ *and*

$$\int_\Omega |\nabla u|^p dx \geq \int_\Omega |\nabla u^\sharp|^p dx. \tag{24}$$

Furthermore, if equality holds in the latter inequality, and

$$|\{x \in B : \nabla u^\sharp(x) = 0\}| = 0$$

then $u(x) = u^\sharp(x)$ *in* Ω.

Proof. See, for example, [11]. Note that, inequality (i) is often proved for non negative functions. However, replacing f by $f + M$ and g by $g + M$ with a suitable constant M, one gets the result for bounded functions. □

Theorem 3 *Let B be a ball centered in the origin, and let g_0 be a bounded function in B. Let $\mathcal{G}$ be the class of rearrangements generated by g_0, and let $J(g)$ be defined as in* (6). *Then, for all $g \in \mathcal{G}$ we have*
1) $J(g) \leq J(g^\sharp)$; *furthermore, if $J(g) = J(g^\sharp) > 0$ then $g(x) = g^\sharp(x)$ a.e. in B.*
2) *If $\int_B g_0(x)dx \geq 0$ then $J(g) \geq J(\tilde{g})$, where $\tilde{g} = (1 - \chi_{\tilde{B}})g_\sharp$, $\tilde{B}$ being a ball concentric with B and such that $\int_{\tilde{B}} g_\sharp\, dx = 0$; if $J(g) = J(\tilde{g})$ then $g = \tilde{g}$ a.e. in B.*

Proof. The proof is essentially the same as that of Theorem 3 of [3], where $p = 2$. For general p, if $g \in \mathcal{G}$ and if u_g is the corresponding eigenfunction, we have

$$J(g) = \frac{\int_B g\, u_g^p dx}{\int_B |\nabla u_g|^p dx}. \tag{25}$$

Since $(u_g^p)^\sharp = (u_g^\sharp)^p$, by the right hand side of (23) we have

$$\int_B g\, u_g^p\, dx \leq \int_B g^\sharp\, (u_g^\sharp)^p dx.$$

Using the latter inequality and (24), from (25) we find

$$J(g) \leq \frac{\int_B g^\sharp\, (u_g^\sharp)^p dx}{\int_B |\nabla u_g^\sharp|^p dx}. \tag{26}$$

Note that $u_g^\sharp \geq 0$ and $u_g^\sharp \in H_0^{1,p}(\Omega)$. Hence, recalling the variational characterization of the maximizer $u_{g^\sharp}$, by (26) we find

$$J(g) \leq \frac{\int_B g^\sharp\, (u_{g^\sharp})^p dx}{\int_B |\nabla u_{g^\sharp}|^p dx} = J(g^\sharp). \tag{27}$$

Now, let $J(g) = J(g^\sharp) > 0$. From (26) and (27) we get

$$\frac{\int_B g\, u_g^p dx}{\int_B |\nabla u_g|^p dx} = \frac{\int_B g^\sharp\, (u_g^\sharp)^p dx}{\int_B |\nabla u_g^\sharp|^p dx} = \frac{\int_B g^\sharp\, (u_{g^\sharp})^p dx}{\int_B |\nabla u_{g^\sharp}|^p dx}.$$

The latter equation together with (23) and (24) yield

$$\int_B |\nabla u_g|^p dx = \int_B |\nabla u_g^\sharp|^p dx. \tag{28}$$

Furthermore, by the variational characterization of the maximizer $u_{g^\sharp}$ and by its uniqueness we must have $u_g^\sharp = u_{g^\sharp}$.

The function $u_{g^\sharp}$ satisfies

$$-\Delta_p u_{g^\sharp} = \frac{1}{\sqrt{J(g^\sharp)}} g^\sharp\, u_{g^\sharp}^{p-1} \ \text{ in } B, \ \ u_{g^\sharp} = 0 \ \text{ on } \partial B. \tag{29}$$

Since $u_{g^\sharp} = u_g^\sharp$, $u_{g^\sharp}$ is radially symmetric, positive and decreasing. With $v(r) = u_{g^\sharp}$ and $z(r) = \frac{1}{\sqrt{J(g^\sharp)}} g^\sharp$ for $|x| = r$, equation (29) can be rewritten as

$$(r^{N-1}(-v')^{p-1})' = r^{N-1} z(r)(v(r))^{p-1}.$$

Integration over $(0, r)$ yields

$$r^{N-1}(-v')^{p-1} = \int_0^r t^{N-1} z(t)(v(t))^{p-1} dt.$$

Recall that $z(r)$ is decreasing and positive near $r = 0$. Hence, $z(r)(v(r))^{p-1}$ is strictly positive near $r = 0$. Therefore, from the previous equation, we find that $v'(r) < 0$ for $0 < r < r_0$, with $r_0 \le R$. We claim that $r_0 = R$. By contradiction, let $r_0 < R$, $v(r_0) > 0$ and $v'(r_0) = 0$. This is possible only if $z(r_0) < 0$. Then, since $z(r)$ is decreasing, we have $z(r) < 0$ on (r_0, R). It follows that $z(r)(v(r))^{p-1} < 0$ on (r_0, R), and that $v'(r) > 0$ there. This cannot happen, since $v(r)$ is decreasing and $v(R) = 0$. Therefore, $v'(r)$ vanishes only at $r = 0$, and $\nabla u_{g^\sharp} = \nabla u_g^\sharp$ vanishes only at the origin. Hence, by (28) and Lemma 7 (ii), we have $u_g = u_g^\sharp = u_{g^\sharp}$.

The functions u_g and $u_{g^\sharp}$ satisfy

$$-\Delta_p u_g = \frac{1}{\sqrt{J(g)}} g u_g^{p-1}, \quad -\Delta_p u_{g^\sharp} = \frac{1}{\sqrt{J(g^\sharp)}} g^\sharp (u_{g^\sharp})^{p-1}.$$

Hence, since $J(g) = J(g^\sharp)$ and $u_g = u_{g^\sharp} > 0$ a.e. in Ω we must have $g = g^\sharp$ almost everywhere in B. Part 1) of the theorem is proved.

Now, assume $\int_B g_0(x)dx > 0$. To prove that $\tilde{g} = (1 - \chi_{\tilde{B}})g_\sharp$ is a minimizer of $J(g)$, let $u_{\tilde{g}}$ be the maximizer of

$$w \mapsto \frac{\int_B \tilde{g}|w|^p dx}{\int_B |\nabla w|^p dx}.$$

Since $\tilde{g}(x) > 0$ in a set of positive measure, we have $J(\tilde{g}) > 0$ and

$$J(u_{\tilde{g}}) = \frac{\int_B \tilde{g} u_{\tilde{g}}^p dx}{\int_B |\nabla u_{\tilde{g}}|^p dx}.$$

Moreover,

$$-\Delta_p u_{\tilde{g}} = \frac{1}{\sqrt{J(\tilde{g})}} \tilde{g}\, u_{\tilde{g}}^{p-1} \ \text{ in } B, \ \ u_{\tilde{g}} = 0 \ \text{ on } \partial B. \tag{30}$$

We have $u_{\tilde{g}}(x) \ge 0$ and, by uniqueness, this function is radially symmetric. If we rewrite equation (30) in radial coordinates (putting $u_{\tilde{g}}(x) = v(r)$ for $|x| = r$) we have

$$-(r^{N-1}|v'|^{p-2}v')' = r^{N-1} \frac{1}{\sqrt{J(\tilde{g})}} \tilde{g}\,(v(r))^{p-1}.$$

From this ordinary differential equation we see that $u_{\tilde{g}}$ decreases as $|x|$ increases, therefore $u_{\tilde{g}} = u_{\tilde{g}}^\sharp$. Hence, using the left hand side of (23) we have

$$\int_B g_\sharp u_{\tilde{g}}^p \, dx \le \int_B g u_{\tilde{g}}^p \, dx. \tag{31}$$

We claim that

$$\int_B g_\sharp u_{\tilde g}^p\,dx = \int_{\tilde B} g_\sharp u_{\tilde g}^p\,dx + \int_{B\setminus\tilde B} g_\sharp u_{\tilde g}^p\,dx = \int_B \tilde g u_{\tilde g}^p\,dx. \tag{32}$$

Indeed, since by (30) $\Delta_p u_{\tilde g} = 0$ in $\tilde B$, we have $u_{\tilde g} = C$ on $\tilde B$ for some constant C. Furthermore, since the integral of $g_\sharp$ over $\tilde B$ vanishes by assumption, we have

$$\int_{\tilde B} g_\sharp u_{\tilde g}^p\,dx = C^p \int_{\tilde B} g_\sharp dx = 0.$$

Finally, since, $g_\sharp = \tilde g$ on $B \setminus \tilde B$, and since $\tilde g = 0$ on $\tilde B$, the claim follows.

Equations (31) and (32) yield

$$\int_B \tilde g u_{\tilde g}^p\,dx \le \int_B g u_{\tilde g}^p\,dx.$$

If $g \in \mathcal{G}$, using the variational characterization of u_g together with the latter inequality, we find

$$J(g) = \frac{\int_B g\, u_g^p dx}{\int_B |\nabla u_g|^p dx} \ge \frac{\int_B g\, u_{\tilde g}^p dx}{\int_B |\nabla u_{\tilde g}|^p dx} \ge \frac{\int_B \tilde g\, u_{\tilde g}^p dx}{\int_B |\nabla u_{\tilde g}|^p dx} = J(\tilde g).$$

We have proved that $\tilde g$ is a minimizer of $J(g)$. Note that $\tilde g$ is the unique minimizer by the strict convexity of $J(g)$ (proved in Theorem 1).

If $\int_B g_0(x)dx = 0$, we have $\tilde B = B$ and $\tilde g = 0$, therefore, the theorem holds trivially in this case. If $J(g) = J(\tilde g)$ then $g = \tilde g$ a.e. in B by the uniqueness of the minimizer. The proof of the theorem is complete. □

Remark. In case $g_0(x) \ge 0$, the set $\tilde B$ in Theorem 3 is empty, and the minimizer $\tilde g$ is the increasing rearrangement $g_\sharp \in \mathcal{G}$. If g_0 takes positive and negative values in a subset of positive measure, the minimizer $\tilde g$ is not in $\mathcal{G}$, because $\tilde g \ge 0$. If $\int_B g_0(x)\,dx < 0$ then any $\underline{g} \in \overline{\mathcal{G}}$ with $\underline{g} \le 0$ is a minimizer of $J(g)$ and $J(\underline{g}) = 0$.

4 Conclusion

In case of $p = 2$, the biological implications of Theorems 2 and 3 are the following. To optimize the persistence of the population, one must locate the favorable resources (such as food, water, refuges, etc.) as far as possible from the boundary of the region, and the unfavorable resources (if any) in a neighborhood of the boundary. Indeed, this configuration maximizes $J(g)$ (and minimizes the principal eigenvalue λ_g).

The situation concerning the extinction of the population is more subtle. In case of absence of unfavorable resources, to optimize the extinction we must locate the favorable resources in a neighborhood of the boundary of Ω. In case we have both favorable and unfavorable resources, it is necessary to compare the amount of each of them. If the favorable resources are prevalent, one should compensate the amount of unfavorable resources by the same amount of favorable ones (this means the one should avoid the use of these two equivalent resources). The remaining part of favorable resources must be located in a neighborhood of the boundary. Indeed, this configuration minimizes $J(g)$ (and maximizes λ_g). If the unfavorable resources are prevalent, there is an arrangement of the resources so that $J(\breve g) = 0$, which corresponds to $\lambda_{\breve g} = \infty$ (we have extinction of the population).

Our results depend strongly on the assumption that the exterior of Ω is hostile, which corresponds to imposing a Dirichlet boundary condition. If the region Ω is closed in the sense that individuals living in Ω never cross the boundary (no-flux condition) then we have Neumann boundary conditions, and the conclusions are quite different from those in the Dirichlet case. For example, in case of $N = 1$, $\Omega = (a, b)$ and $\int_a^b g(x)dx < 0$, we have two minimizers, namely, the decreasing rearrangement and the increasing rearrangement of g. For a proof of this results and other results in case of Neumann boundary conditions, we refer to [12].

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