Boll. Accademia Gioenia di Scienze Naturali - Catania

BOLLAG Vol. 46, N. 376 (2013), pp. 57 - 76
ISSN 0393-7143

Anno di fondazione 1824

A note on Busemann equation†

G. Talenti [1]*

[1] Accademia Gioenia di Scienze Naturali, Catania, Italy

Summary

A second-order quasi-linear partial differential equation of mixed elliptic-hyperbolic type, which both mimics one introduced by A. Busemann in gas dynamics and arises in the study of Minkowski spaces, is considered.

Key words: *Second-order quasi-linear partial differential equations of mixed type, Bäcklund transformations, minimal surfaces, space-like maximal surfaces in Minkowski space, D'Alembert equation, initial value problems.*

Riassunto

In questa comunicazione discutiamo un'equazione differenziale alle derivate parziali del second'ordine, quasi-lineare e di tipo misto ellittico-iperbolico, che imita un'altra introdotta da A. Busemann nella dinamica dei gas e si presenta anche nella teoria degli spazi di Minkowski.

Parole chiave: *Equazioni differenziali alle derivate parziali del second'ordine e del tipo misto ellittico-iperbolico, trasformazioni di Bäcklund, superfici minime, superfici massime nello spazio di Minkowski, equazione di D'Alembert, problemi con valori iniziali.*

1 Introduction

1.1 Background

Consider the three-dimensional, steady, irrotational flow of a perfect gas. Let x, y, z = space rectangular coordinates, φ = velocity potential, σ = sound speed. Standard principles of fluid dynamics (i.e. the equation of continuity, the Euler equations of motion, an equation of state and Bernoullis law) yield the following equations — see e.g. [14], [37], [48]. First,

$$\sigma = \text{ a suitable function of } \nabla\varphi$$

† Lecture given on the occasion of the 70th birthday of Mario Marino, 3-4 May 2013, Catania

* e-mail: giorgiotalenti@gmail.com

— for instance,

$$\sigma(\nabla\varphi) = A\sqrt{B^2 - \varphi_x^2 - \varphi_y^2 - \varphi_z^2} \qquad (A, B = \text{Constants})$$

in the case where the flow is adiabatic and isentropic. Second,

$$\begin{aligned} \left[\sigma^2(\nabla\varphi) - \varphi_x^2\right]\cdot\varphi_{xx} + \left[\sigma^2(\nabla\varphi) - \varphi_y^2\right]\cdot\varphi_{yy} + \left[\sigma^2(\nabla\varphi) - \varphi_z^2\right]\cdot\varphi_{zz} \quad & + \\ -2\varphi_x\varphi_y\cdot\varphi_{xy} - 2\varphi_x\varphi_z\cdot\varphi_{xz} - 2\varphi_y\varphi_z\cdot\varphi_{yz} \quad & = \quad 0 \end{aligned}$$

— a quasi-linear partial differential equation governing φ.

According to works [15] and [16] by Buseman[1], the flow is *conical* if the set of its streamlines[2] is invariant under homothetic transformations. Equivalently, the flow is conical if all its isoclines[3] are rays from the origin. As is easy to see, an irrotational flow is conical if and only if its velocity potential φ obeys

$$\left(x\varphi_{xx} + y\varphi_{xy} + z\varphi_{xz}\right) : \varphi_x = \left(x\varphi_{xy} + y\varphi_{yy} + z\varphi_{yz}\right) : \varphi_y = \left(x\varphi_{xz} + y\varphi_{yz} + z\varphi_{zz}\right) : \varphi_z$$

— i.e. the first-order derivatives of φ are homogeneous functions of x, y, z and all have the same degree.

Busemann showed that if the flow is conical and u, v, w denote the components of the velocity, i.e.

$$u = \varphi_x, \qquad v = \varphi_y, \qquad w = \varphi_z,$$

then the following holds. First, the Jacobian determinant of u, v, w vanishes identically — any component of the velocity is a function of the remaining two. Second, the equation governing w as a function of u and v takes the form

$$\begin{aligned} &\left[1 - \frac{v^2}{\sigma^2} - 2\frac{vw}{\sigma^2}\frac{\partial w}{\partial v} + \left(1 - \frac{w^2}{\sigma^2}\right)\left(\frac{\partial w}{\partial v}\right)^2\right]\cdot\frac{\partial^2 w}{\partial u^2} + \\ 2\;&\left[\frac{uv}{\sigma^2} + \frac{vw}{\sigma^2}\frac{\partial w}{\partial u} + \frac{uw}{\sigma^2}\frac{\partial w}{\partial v} - \left(1 - \frac{w^2}{\sigma^2}\right)\frac{\partial w}{\partial u}\frac{\partial w}{\partial v}\right]\cdot\frac{\partial^2 w}{\partial u\partial v} + \\ &\left[1 - \frac{u^2}{\sigma^2} - 2\frac{uw}{\sigma^2}\frac{\partial w}{\partial u} + \left(1 - \frac{w^2}{\sigma^2}\right)\left(\frac{\partial w}{\partial u}\right)^2\right]\cdot\frac{\partial^2 w}{\partial v^2} = 0. \end{aligned}$$

Aerodynamicists often lower the number of independent variables by virtue of geometric or physical hypotheses. The conical-flow analysis of Busemann provides a mean of descending from a three-dimensional to a two-dimensional potential equation. Several authors put conical flows to use, e.g. Germain [28], [29].

Let us call *dimensional analysis* into play, and *zoom in*. Let h and k be constants obeying

$$h > \sigma(0,0,h), \qquad k^2\cdot\left(\frac{k^2}{\sigma^2(0,0,h)} - 1\right) = 1$$

[1]Adolf Busemann (Lübeck 1901, Boulder 1986) was an eminent aerospace engineer and applied mathematician, and a pioneer of supersonic aerodynamics. He designed the Busemann biplane, which emits no sonic shock waves, and invented the swept wing equipping most modern aircrafts.

[2]Streamlines = trajectories of the velocity field = orbits of

$$dx : \varphi_x(x,y,z) = dy : \varphi_y(x,y,z) = dz : \varphi_z(x,y,z).$$

[3]Isoclines = paths along which the velocity field keeps a constant direction = orbits of

$$\left(\varphi_{xx}dx + \varphi_{xy}dy + \varphi_{xz}dz\right) : \varphi_x = \left(\varphi_{xy}dx + \varphi_{yy}dy + \varphi_{yz}dz\right) : \varphi_y = \left(\varphi_{xz}dx + \varphi_{yz}dy + \varphi_{zz}dz\right) : \varphi_z.$$

— h is the third component of a supersonic velocity, k is a normalizing factor. Replacing

$$u, v, w$$

respectively by

$$\varepsilon \cdot u, \varepsilon \cdot v, h + \varepsilon \cdot k \cdot w$$

in the equation above, and letting

ε approach 0,

result in

$$\left[1 - \left(\frac{\partial w}{\partial v}\right)^2\right] \cdot \frac{\partial^2 w}{\partial u^2} + 2\frac{\partial w}{\partial v}\frac{\partial w}{\partial u} \cdot \frac{\partial^2 w}{\partial u \partial v} + \left[1 - \left(\frac{\partial w}{\partial u}\right)^2\right] \cdot \frac{\partial^2 w}{\partial v^2} = 0$$

— a *toy version* of the full Busemann equation.

1.2 Subject

Motivated by the foregoing arguments, but loyal to more habitual notations, in the present report we comment on the following equation

$$(u_y^2 - 1) \cdot u_{xx} - 2u_y u_x \cdot u_{xy} + (u_x^2 - 1) \cdot u_{yy} = 0, \tag{1}$$

where x and y denote the independent variables and u stands for a real-valued function of x and y. Equation (1) has a *mixed elliptic-hyperbolic* character. Since the coefficient matrix

$$\begin{bmatrix} u_y^2 - 1 & -u_x u_y \\ -u_x u_y & u_x^2 - 1 \end{bmatrix}$$

causes eigenvalues to equal (-1) and $(u_x^2 + u_y^2 - 1)$, a solution u to (1) is *elliptic* in any region where

$$u_x^2 + u_y^2 < 1,$$

and is *hyperbolic* where

$$u_x^2 + u_y^2 > 1.$$

For instance, the formulas

$$u(x, y) = \log\left(\sqrt{x^2 + y^2} + \sqrt{1 + x^2 + y^2}\right), \qquad u(x, y) = \arcsin\sqrt{x^2 + y^2},$$

represent an everywhere elliptic solution and an everywhere hyperbolic solution to equation (1), respectively. The formula

$$u(x, y) = \log\left(\frac{\cosh x}{\cosh y}\right)$$

supplies a solution to (1) whose streamlines obey

$$\sinh|x| \cdot \sinh|y| = \text{Constant},$$

and whose type changes — elliptic in the region where

$$\sinh|x| \cdot \sinh|y| < 1,$$

hyperbolic where

$$\sinh|x| \cdot \sinh|y| > 1.$$

2 Digressing on Bäcklund transformations

2.1 General

Loosely speaking, a Bäcklund transformation converts a solution to some partial differential equation into a different solution to the same equation, or into a solution to another partial differential equation. Such a transformation allows an extra solution to a partial differential equation to come out if one particular solution to the same or another equation is in hand. A Bäcklund transformation typically looks like a first-order partial differential system, which relates two functions in a convenient way and drives them to obey partial differential equations individually. Bäcklund transformations may be of considerable service; however, no systematic way of finding them is available. These transformations trace back to works by L. Bianchi and A.V. Bäcklund in differential geometry, and play a role especially in soliton theory and integrable systems. They come up in gas dynamics too, and are a key to the present work. Relevant references include [3], [24], [44], [45], [46], [51].

2.2 Specimens

(i) If $a_{11}, a_{12}, a_{21}, a_{22}$ are tractable functions of x and y and

$$a_{11}a_{22} - a_{12}a_{21} \neq 0$$

everywhere, then the transformation attached to the following formula

$$\nabla v = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \nabla u$$

generalizes *Cauchy-Riemann equations*. It turns any suitably smooth solution of

$$\operatorname{div}\left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \nabla u \right\} = 0,$$

— a second-order linear partial differential equation in divergence form — into a solution of

$$\operatorname{div}\left\{ \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T \nabla v \right\} = 0$$

which obeys the orthogonality condition

$$\text{inner product of } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \nabla u \text{ and } \nabla v = 0.$$

(ii) Let a_{11}, a_{12}, a_{22} be bounded, measurable functions of x and y such that

$$a_{11} \neq 0, \qquad a_{22} \neq 0;$$

let φ obey

$$a_{11}\varphi_{xx} + 2a_{12}\varphi_{xy} + a_{22}\varphi_{yy} = 0$$

— a second-order linear partial differential equation in non-divergence form. If

$$\begin{bmatrix} u \\ v \end{bmatrix} = \nabla \varphi,$$

then u and v are the images of one another under the following transformation

$$\nabla v = -\frac{1}{a_{22}}\begin{bmatrix} 0 & -a_{22} \\ a_{11} & 2a_{12} \end{bmatrix}\nabla u, \qquad \nabla u = -\frac{1}{a_{11}}\begin{bmatrix} 2a_{12} & a_{22} \\ -a_{11} & 0 \end{bmatrix}\nabla v,$$

and obey

$$\operatorname{div}\left\{\frac{1}{a_{22}}\begin{bmatrix} a_{11} & 2a_{12} \\ 0 & a_{22} \end{bmatrix}\nabla u\right\} = 0, \qquad \operatorname{div}\left\{\frac{1}{a_{11}}\begin{bmatrix} a_{11} & 0 \\ 2a_{12} & a_{22} \end{bmatrix}\nabla v\right\} = 0.$$

(iii) Suppose $0 \leq \varrho \mapsto j(\varrho)$ is a smooth real-valued function, whose derivative vanishes at zero. Let u be an extremal of the variational integral

$$\iint j\left(\sqrt{u_x^2 + u_y^2}\right) dx\, dy,$$

i.e. a sufficiently smooth solution to

$$\frac{\partial}{\partial x}\left\{\frac{dj}{d\varrho}\left(\sqrt{u_x^2+u_y^2}\right)\frac{u_x}{\sqrt{u_x^2+u_y^2}}\right\} + \frac{\partial}{\partial y}\left\{\frac{dj}{d\varrho}\left(\sqrt{u_x^2+u_y^2}\right)\frac{u_y}{\sqrt{u_x^2+u_y^2}}\right\} = 0.$$

The *hodograph polar coordinates* ϱ and ω given by

$$\nabla u = \varrho \cdot \begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix},$$

i.e. the *length* and the *azimuth* of the gradient, are related by a Bäcklund transformation, namely

$$\nabla\omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix}\begin{bmatrix} \varrho j''(\varrho)/j'(\varrho) & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}\frac{\nabla\varrho}{\varrho}.$$

For instance, any solution u to equation (1) satisfies

$$\begin{aligned} \frac{1}{\varrho}\frac{\partial\varrho}{\partial x} + \frac{\partial\omega}{\partial y} : \varrho^2\cos\omega &= \frac{1}{\varrho}\frac{\partial\varrho}{\partial y} - \frac{\partial\omega}{\partial x} : \varrho^2\sin\omega \\ &= -\sin\omega\frac{\partial\omega}{\partial x} + \cos\omega\frac{\partial\omega}{\partial y}. \end{aligned}$$

(iv) The transformation attached to the formula

$$u = \log\left(2\frac{v_x^2 + v_y^2}{v^2}\right)$$

maps solutions to

$$\Delta v = 0$$

(*Laplace equation*) into solutions to

$$\Delta u = \exp(u)$$

(*Liouville equation*).

(v) The transformations attached to the formulas

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \left(1 + v^2\right)^{-1/2} \begin{bmatrix} 1 \\ v \end{bmatrix}, \qquad v = \frac{u_y}{u_x},$$

are the inverse of one another. They convert any suitably smooth solution u to

$$u_x^2 + u_y^2 = 1$$

(a prototypal *eikonal equation* of geometrical optics) into a solution v to

$$v_x + v v_y = 0$$

(*inviscid Burgers equation*), and vice versa — the level-lines and the shock-line of v are the isoclines and the caustic of u, respectively.

(vi) Let ε be a positive constant parameter. The transformations attached to the formulas

$$v_x = -\tfrac{1}{2\varepsilon} u v, \qquad v_y = -\tfrac{1}{2}\left(u_x - \tfrac{1}{2\varepsilon} u^2\right) v,$$

$$u = -2\varepsilon \tfrac{v_x}{v}$$

appeared first in [20] and [32] and are currently known as *Cole-Hopf transformations*. They are the inverse of one another, and map solutions to

$$u_y + u u_x = \varepsilon \cdot u_{xx}$$

(*viscous Burgers equation*) into solutions to

$$v_y = \varepsilon \cdot v_{xx}$$

(*heat equation*), and vice versa.

(vii) If p is a constant parameter such that $1 < p < \infty$, the following recipe

$$\nabla v = |\nabla u|^{p-2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \nabla u, \qquad \nabla u = |\nabla u|^{p/(p-1)-2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \nabla v,$$

defines transformations that are the inverse of one another. They map any solution u of

$$\left[(p-1)u_x^2 + u_y^2\right] \cdot u_{xx} + 2(p-2)u_x u_y \cdot u_{xy} + \left[u_x^2 + (p-1)u_y^2\right] \cdot u_{yy} = 0$$

(*p-Laplace equation*) into a solution v of

$$\left[v_x^2 + (p-1)v_y^2\right] \cdot v_{xx} - 2(p-2)v_x v_y \cdot v_{xy} + \left[(p-1)v_v^2 + v_y^2\right] \cdot u_{yy} = 0$$

($p/(p-1)$-*Laplace equation*) in such a way that

$$u_x v_x + u_y v_y = 0.$$

Relevant information is in [4], [5], [6] and [7].

3 Maximal space-like surfaces in Minkowski space

3.1 General

Elliptic solutions to (1) obey both

$$u_x^2 + u_y^2 < 1,$$

and

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1-u_x^2-u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1-u_x^2-u_y^2}}\right) = 0.$$

Recall that the equations

$$\text{metric} = (dx)^2 + (dy)^2 - (du)^2,$$
$$\text{area of a space-like graph } u = \iint \sqrt{1-u_x^2-u_y^2}\, dx\, dy,$$

hold in the three-dimensional Minkowski space. Therefore, the elliptic solutions to (1) render an appropriate area a *maximum* — they represent *space-like maximal surfaces* in the three-dimensional Minkowski space.

Selected apropos references include [1], [2], [8], [9], [10], [11], [12], [13], [17], [18], [19], [25], [26], [27], [33], [34], [39], [40], [38], [41], [47], [49], [50].

3.2 Formulas

Because of the previous observation, *elliptic solutions* to equation (1) can be parametrically represented by Kobayashi formulas [34]:

$$\begin{aligned} \lambda, \mu &= \text{real parameters}, \\ x &= \mathrm{Re}\frac{1}{2}\int^{\lambda+i\mu} f(\zeta)\left[1+g(\zeta)^2\right] d\zeta, \\ y &= \mathrm{Re}\frac{i}{2}\int^{\lambda+i\mu} f(\zeta)\left[1-g(\zeta)^2\right] d\zeta, \\ u &= -\mathrm{Re}\int^{\lambda+i\mu} f(\zeta)g(\zeta)\, d\zeta, \end{aligned} \tag{2}$$

— here f is holomorphic, g is a meromorphic function such that $|g| \neq 1$ and $f \cdot g^2$ is holomorphic.

For example, putting $f(\zeta) = 6(1+\zeta)^2$ and $g(\zeta) = (1-\zeta)/(1+\zeta)$ into Kobayashi formulas results in the quartic equation

$$(x-u)^4 = 27(x^2+y^2-u^2),$$

which supplies an elliptic solution to (1) and whose graph is shown in Figure 1.

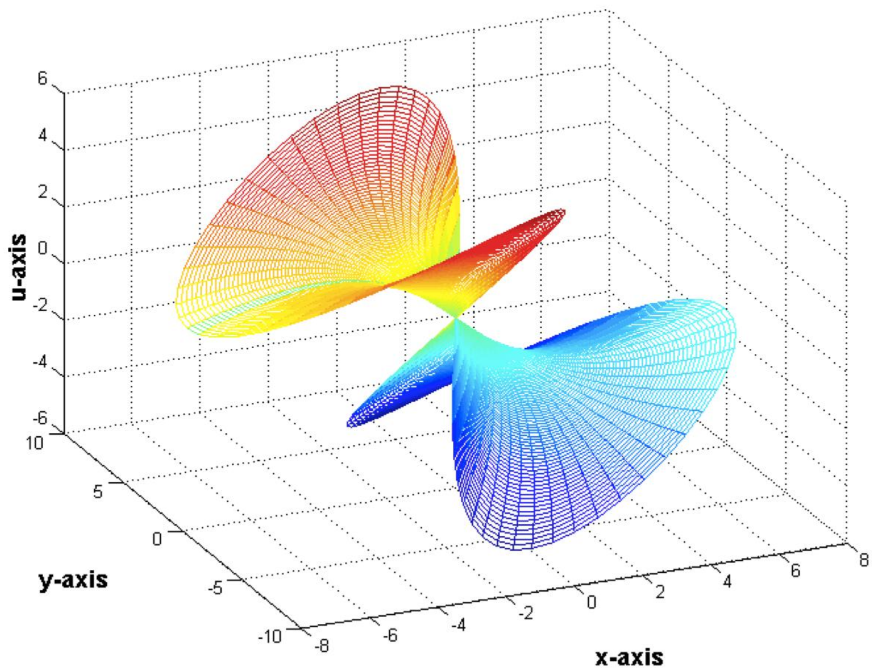

Figure 1: An elliptic solution to equation (1).

3.3 Allied minimal surfaces

Elliptic solutions to equation (1) are closely related to non-parametric *minimal surfaces* in Euclidean three-dimensional space, whose theory can be found in such books as [21], [23], [42], [43]. Here is a reason: *apposite Bäcklund transformations convert the objects in question into each other*. These transformations allow statements on elliptic solutions of (1) to follow automatically from properties of minimal surfaces [4]. In particular, they put maximal surfaces from Minkowski three-dimensional space and customary minimal surfaces on the same footing.

The Bäcklund transformations attached to the following equations

$$\nabla v = \frac{1}{\sqrt{1-|\nabla u|^2}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \nabla u, \qquad \nabla u = \frac{1}{\sqrt{1+|\nabla v|^2}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \nabla v, \tag{3}$$

amount to rotating gradients by ninety degrees, then stretching them suitably. They are the inverse of one another, and enjoy properties (i) to (iii) listed below.

(i) The former acts on elliptic solutions u to equation (1) — such solutions are just what make

$$\frac{1}{\sqrt{1-|\nabla u|^2}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \nabla u,$$

well-defined and a gradient.

(ii) The latter acts on solutions v to either

$$\frac{\partial}{\partial x}\left(\frac{v_x}{\sqrt{1+v_x^2+v_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{v_y}{\sqrt{1+v_x^2+v_y^2}}\right) = 0.$$

[4]For instance, Bernstein-Calabi theorem on elliptic entire solutions to equation (1) is demonstrated in [47] along this line.

or

$$(v_y^2+1)\cdot v_{xx}-2v_yv_x\cdot v_{xy}+(v_x^2+1)\cdot v_{yy}=0,$$

i.e. on functions whose graphs are *minimal surfaces*. Such functions are precisely what make

$$\frac{1}{\sqrt{1+|\nabla v|^2}}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\cdot\nabla v,$$

a gradient.

(iii) Both turn any elliptic solution to (1) into a non-parametric minimal surface, and simultaneously turn any non-parametric minimal surface from Euclidean three-dimensional space into an elliptic solution to (1).

Either one of equations (3) implies

$$\left(1-u_x^2-u_y^2\right)\cdot\left(1+v_x^2+v_y^2\right)=1,\qquad u_xv_x+u_yv_y=0 \tag{4}$$

— a special first-order fully non-linear partial differential system having a rotation-invariant structure. Conversely, system (4) plus the condition

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}>0$$

imply both equations (3). The following extra statements ensue.

(iv) The Bäcklund transformations in hand *decouple* solution pairs to system (4).

(v) The *entries* of any solution pair to system (4), whose Jacobian determinant does not change its sign, represent an *elliptic solution* to equation (1) and a standard *minimal surface*, respectively. Thus system (4) *pairs off* elliptic solutions to equation (1) and minimal surfaces.

The notion of *Chaplygin gas* does enter our game. According to usage, "Chaplygin gas" is a nickname for a hypothetical fluid whose adiabatic constant equals (-1), i.e. whose density and pressure are inversely proportional to one another. An aficionado might realize that the following holds if units are appropriate. First: the Bäcklund transformations attached to equations (3) are precisely those relating the *velocity potential* and the *stream function* of a Chaplygin gas. Second: while the minimal surface equation governs the velocity potential, *equation (1) governs the stream function of a Chaplygin gas*.

Here is a sample pair obeying (3):

$$u(x,y)=\arcsin(\sin x\cdot\sin y),\qquad v(x,y)=\log\left(\frac{\cos x}{\cos y}\right)$$

— the former entry is an elliptic solution to (1), the latter is Scherk's minimal surface.

4 Hyperbolic solutions

4.1 Formulas

Hyperbolic solutions to equation (1) result from formulas of Gu and Li, [30], [31] and [36]:

$$\begin{aligned}
\lambda,\mu &= \text{real parameters},\\
x &= \int^{\lambda} a(\lambda)\cos\lambda\,d\lambda+\int^{\mu} b(\mu)\cos\mu\,d\mu,\\
y &= \int^{\lambda} a(\lambda)\sin\lambda\,d\lambda+\int^{\mu} b(\mu)\sin\mu\,d\mu,\\
u &= \int^{\lambda} a(\lambda)\,d\lambda+\int^{\mu} b(\mu)\,d\mu,
\end{aligned} \tag{5}$$

which involve two non-zero real functions a and b at user's disposal, and imply

$$\cos\left(\frac{\lambda-\mu}{2}\right)\cdot u_x = \cos\left(\frac{\lambda+\mu}{2}\right), \qquad \cos\left(\frac{\lambda-\mu}{2}\right)\cdot u_y = \sin\left(\frac{\lambda+\mu}{2}\right),$$

$$2\sin(\lambda-\mu)\cos^2\left(\frac{\lambda-\mu}{2}\right)\cdot\begin{bmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy}\end{bmatrix}$$
$$= \frac{1}{a(\lambda)}\begin{bmatrix}\sin\mu \\ -\cos\mu\end{bmatrix}\cdot\begin{bmatrix}\sin\mu & -\cos\mu\end{bmatrix} - \frac{1}{b(\mu)}\begin{bmatrix}\sin\lambda \\ -\cos\lambda\end{bmatrix}\cdot\begin{bmatrix}\sin\lambda & -\cos\lambda\end{bmatrix}.$$

For instance, coupling Gu's formulas and

$$a(\lambda) = \cos(5\lambda), \qquad b(\mu) = \sin(\sqrt{15}\mu),$$

leads to Figure 2; coupling the same formulas and

$$a(\lambda) = \alpha\sin(5\lambda), \qquad b(\mu) = \frac{\beta}{\sqrt{2\pi\mu}} \qquad (\alpha, \beta = \text{ Constants}),$$

leads to Figure 3.

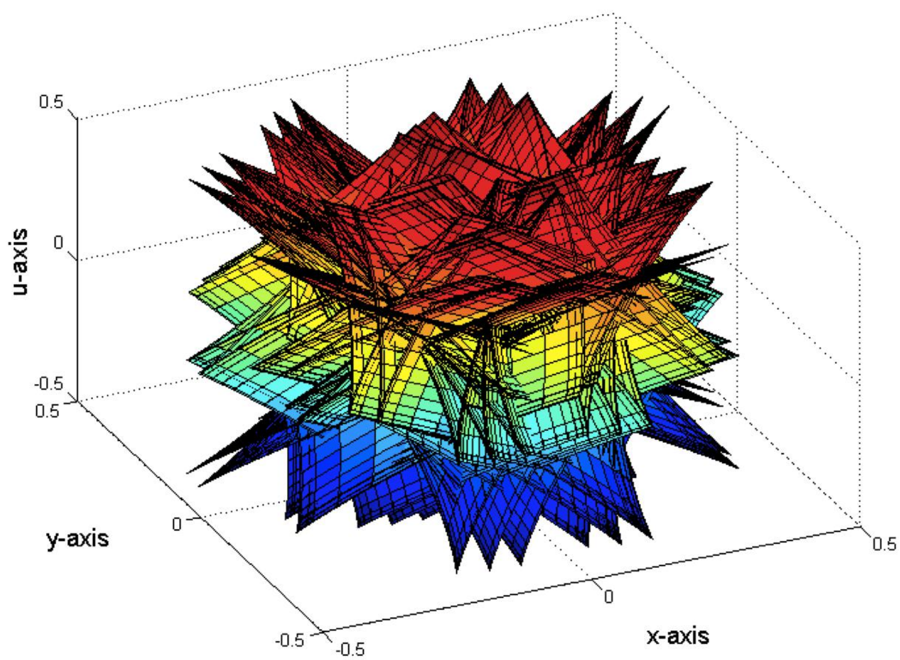

Figure 2: A hyperbolic solution to (1).

4.2 Allied hyperbolic solutions

The Bäcklund transformations attached to the formulas

$$\nabla v = \frac{1}{\sqrt{|\nabla u|^2-1}}\begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix}\cdot\nabla u, \qquad \nabla u = \frac{1}{\sqrt{|\nabla v|^2-1}}\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}\cdot\nabla v, \tag{6}$$

are the inverse of one another, and enjoy properties (i) to (iv) listed below.

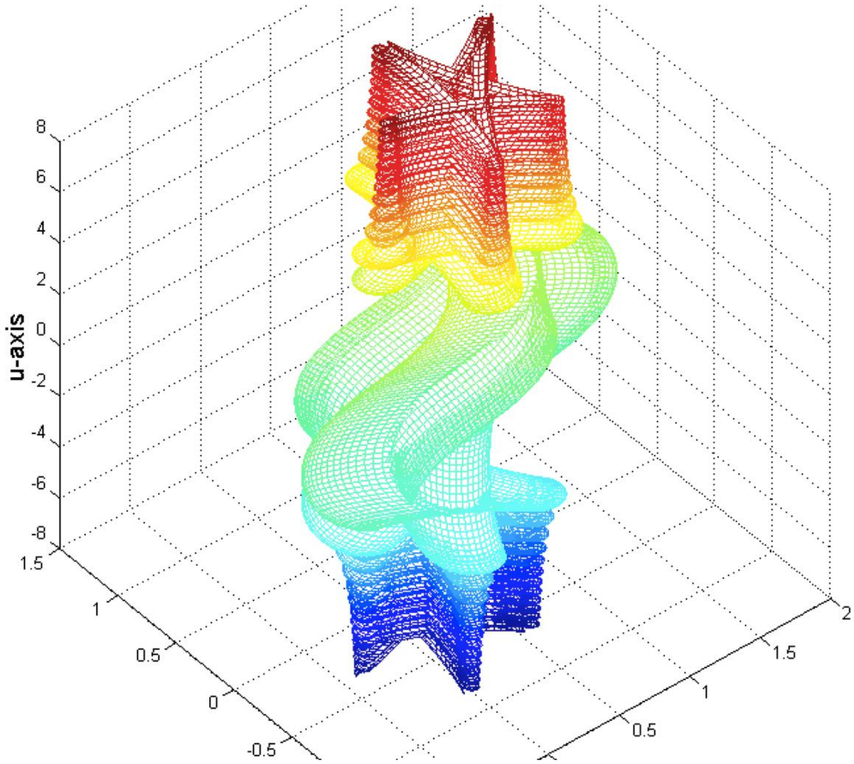

Figure 3: Another hyperbolic solution to (1).

(i) Both act on hyperbolic solutions to equation (1).

(ii) They convert any hyperbolic solution u to equation (1) into another hyperbolic solution v to the same equation.

(iii) They imply

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \frac{u_x^2 + u_y^2}{\sqrt{u_x^2 + u_y^2 - 1}}, \qquad \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \frac{v_x^2 + v_y^2}{\sqrt{v_x^2 + v_y^2 - 1}},$$

and

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \geq 2$$

— in particular, cause u and v to locally become genuine *curvilinear coordinates*.

Either one of equations (6) implies

$$\left(u_x^2 + u_y^2 - 1\right) \cdot \left(v_x^2 + v_y^2 - 1\right) = 1, \qquad u_x v_x + u_y v_y = 0 \tag{7}$$

— a special first-order fully non-linear partial differential system having a rotation-invariant structure. Conversely, system (7) plus the condition

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} > 0$$

imply the two equations (6). The following results. First, the Bäcklund transformations in hand *decouple* solution pairs to system (7). Second, *both entries* of any solution pair to (7), whose Jacobian determinant does not change its sign, satisfy equation (1) — in other words, system (7) *pairs off* solutions to equation (1).

For instance, the formulas

$$u(x,y) = \log\left(\frac{\sinh x}{\cosh y}\right), \qquad v(x,y) = \log\left(\cosh x \cdot \sinh y + \sqrt{1 + \cosh^2 x \cdot \sinh^2 y}\right)$$

supply two allied hyperbolic solutions to (1). The formula

$$u(x,y) = \log\left(\frac{\cosh x}{\cosh y}\right)$$

was met in Section (1.2). It provides a solution to (1) that changes its type and has the following two mates

$$v(x,y) = \log\left(\sinh x \cdot \sinh y + \sqrt{\sinh^2 x \cdot \sinh^2 y - 1}\right), \qquad v(x,y) = \arcsin\left(\sinh x \cdot \sinh y\right)$$

— the former is a pure hyperbolic solution to (1), the latter is a minimal surface.

4.3 D'Alembert equation

Suppose u and v are *hyperbolic solutions* to equation (1), and the foregoing equations (6) pair them off. Here think of u and v as *curvilinear coordinates*, and think of x and y as *functions of* u *and* v — in other words, interchange the role of dependent and independent variables.

The following statements hold.

(i) x and y obey both

$$x_u^2 + y_u^2 \leq 1, \qquad \frac{\partial}{\partial v}\begin{bmatrix} x \\ y \end{bmatrix} = \sqrt{\frac{1}{x_u^2 + y_u^2} - 1}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\frac{\partial}{\partial u}\begin{bmatrix} x \\ y \end{bmatrix} \tag{8}$$

and the following system

$$\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 = 1, \qquad \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} = 0. \tag{9}$$

Incidentally, the latter has a noteworthy geometric significance. Letting E, F and G stand for the coefficients of the Euclidean metric in curvilinear coordinates u and v, i.e.

$$(dx)^2 + (dy)^2 = E\,(du)^2 + 2F\,du\,dv + G\,(dv)^2,$$

allows it to read thus

$$E + G = 1, \qquad F = 0.$$

(ii) x and y obey D'Alembert equation.

(iii) u and v can be represented thus

$$\begin{aligned} u &= (\lambda + \mu)/\sqrt{2}, & v &= (\lambda - \mu)/\sqrt{2}, \\ x &= [A(\lambda) + B(\mu)]/\sqrt{2}, & y &= [C(\lambda) + D(\mu)]/\sqrt{2} \end{aligned} \tag{10}$$

— here

$$\lambda, \mu = \text{parameters}$$

and A, B, C, D satisfy

$$\left[\frac{dA}{d\lambda}(\lambda)\right]^2 + \left[\frac{dC}{d\lambda}(\lambda)\right]^2 = 1, \qquad \left[\frac{dB}{d\mu}(\mu)\right]^2 + \left[\frac{dD}{d\mu}(\mu)\right]^2 = 1,$$

$$\begin{vmatrix} dA(\lambda)/d\lambda & dB(\mu)/d\mu \\ dC(\lambda)/d\lambda & dD(\mu)/d\mu \end{vmatrix} \leq 0.$$

Proof of (i). Coupling

$$\nabla v = \frac{1}{\sqrt{|\nabla u|^2 - 1}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \nabla u,$$

one of equations (6), and

$$\frac{\partial(x,y)}{\partial(u,v)} = \left[\frac{\partial(u,v)}{\partial(x,y)}\right]^{-1},$$

a consequence of the inverse mapping theorem, results in

$$\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \left(u_x^2 + u_y^2\right)^{-1} \begin{bmatrix} u_x & -u_y \\ u_y & u_x \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{u_x^2 + u_y^2 - 1} \end{bmatrix}.$$

Statement (i) follows.

Proof of (ii). Letting

$$D = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix},$$

causes the inverse mapping of

$$(x,y) \mapsto [u(x,y), v(x,y)]$$

to satisfy

$$D \cdot \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} v_y & -u_y \\ -v_x & u_x \end{bmatrix},$$

$$\begin{aligned} D^3 \cdot \begin{bmatrix} x_{uu} & x_{uv} \\ x_{uv} & x_{vv} \end{bmatrix} = \; & -v_y \begin{bmatrix} v_y & -v_x \\ -u_y & u_x \end{bmatrix} \begin{bmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{bmatrix} \begin{bmatrix} v_y & -u_y \\ -v_x & u_x \end{bmatrix} \\ & +u_y \begin{bmatrix} v_y & -v_x \\ -u_y & u_x \end{bmatrix} \begin{bmatrix} v_{xx} & v_{xy} \\ v_{xy} & v_{yy} \end{bmatrix} \begin{bmatrix} v_y & -u_y \\ -v_x & u_x \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} D^3 \cdot \begin{bmatrix} y_{uu} & y_{uv} \\ y_{uv} & y_{vv} \end{bmatrix} = \; & +v_x \begin{bmatrix} v_y & -v_x \\ -u_y & u_x \end{bmatrix} \begin{bmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{bmatrix} \begin{bmatrix} v_y & -u_y \\ -v_x & u_x \end{bmatrix} \\ & -u_x \begin{bmatrix} v_y & -v_x \\ -u_y & u_x \end{bmatrix} \begin{bmatrix} v_{xx} & v_{xy} \\ v_{xy} & v_{yy} \end{bmatrix} \begin{bmatrix} v_y & -u_y \\ -v_x & u_x \end{bmatrix}, \end{aligned}$$

We infer that

$$D^3\left(\frac{\partial^2 x}{\partial u^2}-\frac{\partial^2 x}{\partial v^2}\right) = v_y(v_x^2+v_y^2)\left[(u_y^2-1)u_{xx}-2u_xu_yu_{xy}+(u_x^2-1)u_{yy}\right]+ u_y(u_x^2+u_y^2)\left[(v_y^2-1)v_{xx}-2v_xv_yv_{xy}+(v_x^2-1)v_{yy}\right],$$

$$D^3\left(\frac{\partial^2 y}{\partial u^2}-\frac{\partial^2 y}{\partial v^2}\right) = v_x(v_x^2+v_y^2)\left[(1-u_y^2)u_{xx}+2u_xu_yu_{xy}+(1-u_x^2)u_{yy}\right]+ u_x(u_x^2+u_y^2)\left[(1-v_y^2)v_{xx}+2v_xv_yv_{xy}+(1-v_x^2)v_{yy}\right],$$

and conclude thus

$$\frac{\partial^2 x}{\partial u^2}-\frac{\partial^2 x}{\partial v^2}=0, \qquad \frac{\partial^2 y}{\partial u^2}-\frac{\partial^2 y}{\partial v^2}=0,$$

as claimed.

Proof of (iii). Any solution to D'Alembert equation is the sum of two waves, one progressive and the other regressive. Statement (iii) is therefore a consequence of (i) and (ii).

End of proofs.

5 Initial value problems

Let the following ingredients be in hand. First,

$$x=\alpha(s), \qquad y=\beta(s) \tag{11}$$

— a parametric representation of a plane curve. Think of it as the initial curve; assume smoothness and

$$\left[\frac{d\alpha}{ds}(s)\right]^2+\left[\frac{d\beta}{ds}(s)\right]^2\equiv 1,$$

i.e.

$$s = \text{arclength}.$$

Second,

$$s\mapsto\gamma(s) \tag{12}$$

— a real-valued, smooth function defined on the above curve.

An *initial value problem* consists in demanding that u obeys the following conditions

$$u(x,y)=\gamma(s) \quad \text{and} \quad -u_x(x,y)\frac{d\beta}{ds}(s)+u_y(x,y)\frac{d\alpha}{ds}(s)=0 \tag{13}$$

at every point from the initial curve, and

$$u \text{ obeys equation (1)} \tag{14}$$

in a neighborhood of the initial curve.

As is easy to check, problem (11)-(12)-(13)-(14) is *non-characteristic* if

$$\left|\frac{d\gamma}{ds}(s)\right| \neq 1 \qquad \text{everywhere;} \tag{15}$$

if

$$\left|\frac{d\gamma}{ds}(s)\right| > 1 \qquad \text{everywhere,} \tag{16}$$

then any relevant solution leaves the initial curve in a *hyperbolic status*.

Results from the previous section, plus routine manipulations, lead to the following.

Proposition. If condition (16) is in force, the initial value problem (11)-(12)-(13)-(14) has exactly one hyperbolic solution. Let u be such a solution, and let v be a proper hyperbolic mate — i.e. the Bäcklund transform of u that obeys

$$\nabla v = \frac{1}{\sqrt{|\nabla u|^2 - 1}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \nabla u,$$

and is such that

$$v(x, y) = 0$$

at every point from the initial curve. Then u and v are represented by formulas (10), where A, B, C, D are given by

$$\begin{aligned}
\sqrt{2}\frac{d}{ds}A\left(\frac{\gamma(s)}{\sqrt{2}}\right) &= \frac{d\alpha}{ds}(s) - \sqrt{\left[\frac{d\gamma}{ds}(s)\right]^2 - 1} \cdot \frac{d\beta}{ds}(s), \\
A\left(\frac{\gamma(s)}{\sqrt{2}}\right) + B\left(\frac{\gamma(s)}{\sqrt{2}}\right) &= \sqrt{2}\alpha(s), \\
\sqrt{2}\frac{d}{ds}C\left(\frac{\gamma(s)}{\sqrt{2}}\right) &= \sqrt{\left[\frac{d\gamma}{ds}(s)\right]^2 - 1} \cdot \frac{d\alpha}{ds}(s) + \frac{d\beta}{ds}(s), \\
C\left(\frac{\gamma(s)}{\sqrt{2}}\right) + D\left(\frac{\gamma(s)}{\sqrt{2}}\right) &= \sqrt{2}\beta(s).
\end{aligned} \tag{17}$$

Formulas (17) take the following simpler form

$$\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \dfrac{1}{\sqrt{\kappa^2 - 1}} & -\sqrt{\kappa^2 - 1} \\ & 1 \end{bmatrix} \begin{bmatrix} \alpha\left(\frac{\sqrt{2}}{\kappa}\lambda\right) \\ \beta\left(\frac{\sqrt{2}}{\kappa}\lambda\right) \end{bmatrix},$$

$$\begin{bmatrix} B(\mu) \\ D(\mu) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \dfrac{1}{-\sqrt{\kappa^2 - 1}} & \sqrt{\kappa^2 - 1} \\ & 1 \end{bmatrix} \begin{bmatrix} \alpha\left(\frac{\sqrt{2}}{\kappa}\mu\right) \\ \beta\left(\frac{\sqrt{2}}{\kappa}\mu\right) \end{bmatrix},$$

in the case where

$$\kappa = \text{Constant}, \qquad \gamma(s) = \kappa \cdot s$$

(i.e. the curve in the three-dimensional space, which spans the solution surface, is a *helix*) and moreover

$$|\kappa| > 1.$$

Here is an apropos example.
A *nephroid of Huygens* is the graph of the following equations

$$
\begin{aligned}
x &= (3\cos t - \cos 3t)/12, \\
y &= (3\sin t - \sin 3t)/12, \\
t &= \pi \cdot (\text{integer part of } s) + 2\arcsin\left(\sqrt{\text{fractional part of } s}\right), \\
s &= \text{arc length}
\end{aligned}
$$

— see [35], for instance. Let α and β be specified accordingly, and let γ be specified thus

$$\gamma(s) = 2.0156 \cdot s.$$

Figure 4 shows the nephroid in question, and the relevant helix above it. Figures 5 and 6 show the solution u to problem (11)-(12)-(13)-(14) and its mate v, respectively.

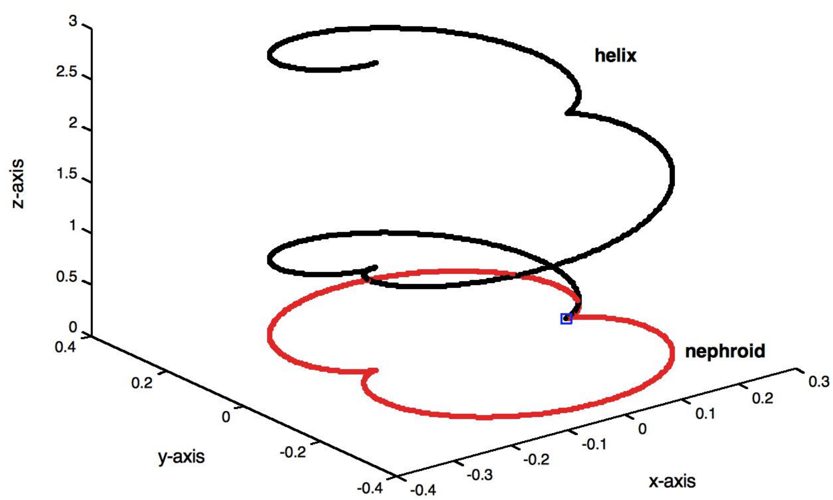

Figure 4: View of the initial nephroid and the helix spanning the solution surface.

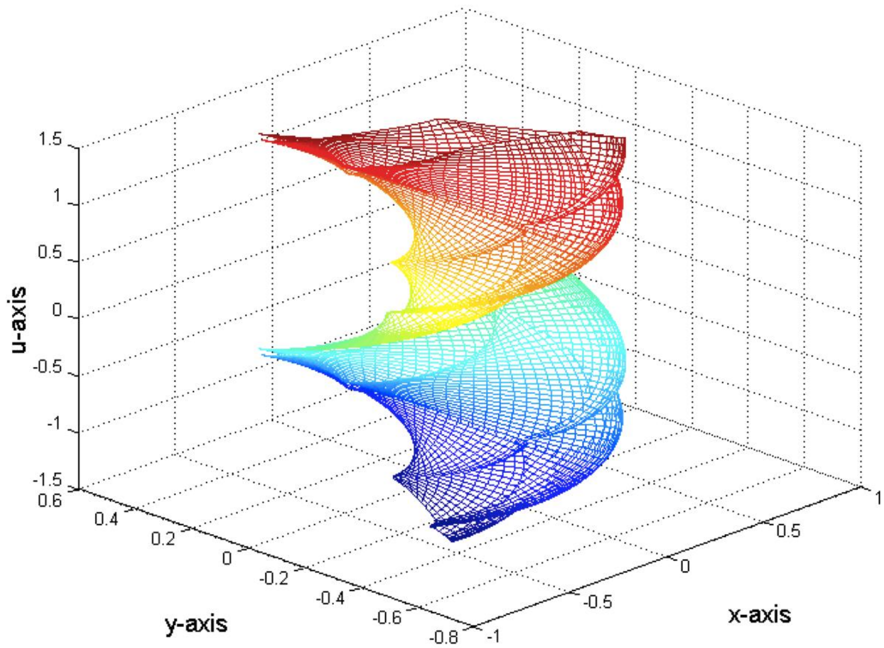

Figure 5: View of a hyperbolic solution u to an initial value problem for equation 1.

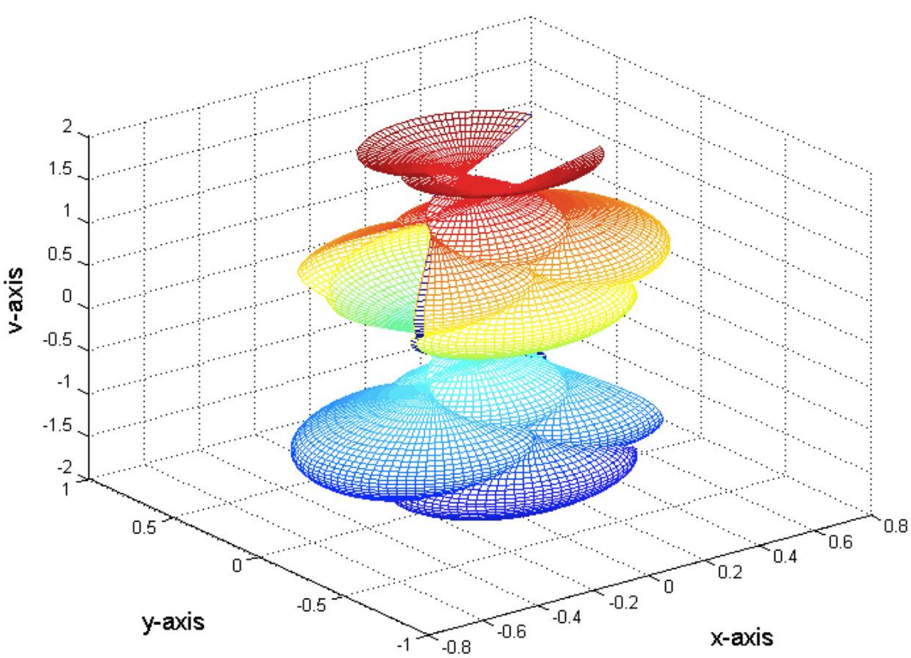

Figure 6: View of a hyperbolic mate v.

References

[1] L.J. Alias & R.M.B. Chaves & P. Mira, Björling problem for maximal surfaces in Lorentz-Minkowski space. Mat. Proc. Cambridge Philos. Soc. 134 (2003) 289-316.

[2] L.J. Alias & B. Palmer, A duality result between the minimal surface equation and the maximal surface equation. An. Acad. Bras. Cienc. 73 (2001) 161-164.

[3] R.L. Anderson & N.H. Ibragimov, Lie-Bäcklund transformations in applications, SIAM (1979).

[4] G. Aronsson, A stream function technique for the p-harmonic equation in the plane. Department of Mathematics, University of Luleå, 1986-3.

[5] G. Aronsson, Representation of a p-harmonic function near a critical point in the plane. Manuscripta Mathematica 66 (1986) 73-95.

[6] G. Aronsson, On certain p-harmonic functions in the plane. Manuscripta Math. 61 (1988) 79-101.

[7] G. Aronsson & P. Lindqvist, On p-harmonic functions in the plane and their stream functions. J. Diff. Equations 74 (1988) 157-178.

[8] R. Bartnik, The existence of maximal surfaces. Miniconference on operator theory and partial differential equations (Canberra, 1983), 47-51, Proc. Centre Math. Anal. Austral. Nat. Univ., 5, Austral. Nat. Univ., Canberra, 1984.

[9] R. Bartnik, Existence of maximal surfaces in asymptotically flat spacetimes. Comm. Math. Phys. 94 (1984) 155-175.

[10] R. Bartnik, Maximal surfaces and general relativity. Miniconference on geometry and partial differential equations 2 (Canberra, 1986), 24-49, Proc. Centre Math. Anal. Austral. Nat. Univ., 12, Austral. Nat. Univ., Canberra, 1987.

[11] R. Bartnik, Regularity of variational maximal surfaces. Acta Math. 166 (1988) 145-181.

[12] R. Bartnik & P.T. Chrusciel & O. Murchadha, On maximal surfaces in asymptotically flat space-time. Comm. Math. Phys. 130 (1990) 95-109.

[13] R. Bartnik & L. Simon, Spacelike hypersurfaces with prescribed boundary values and maen curvature. Comm. Math. Phys. 87 (1982) 131-152.

[14] L. Bers, Mathematical aspects of subsonic and transonic gas dynamics. Surveys in Applied Mathematics 3, John Wiley & Sons, 1958.

[15] A. Busemann, Die achsensymmetrische kegelige Überschallströmung. Luftfahrtforshung 19 (1942) 137-144.

[16] A. Busemann, Infinitesimale kegelige Überschallströmung. Schriften der Deutschen Akademie für Luftfahrtforshung 7B (1943) 105-121.

[17] E. Calabi, Examples of Bernstein problems for some non-linear equations. Proc. Symposia Pure Math. 15 (1970) 223-230.

[18] W. Chen & N. Su, The self-conjugate maximal surfaces in L^3. Northeast. Math. J. 14 (1998) 9-16.

[19] S.Y. Cheng & S.T. Yau, Maximal spacelike surfaces in Lorentz-Minkowski spaces. Ann. Math. 104 (1976) 407-419.

[20] J.D. Cole, On a quasilinear parabolic equation occurring in aerodinamics. Quart. Appl. Math. 9 (1951) 225-236.

[21] R. Courant, Dirichlets principle, conformal mapping, and minimal surfaces. Dover Publications, 1950.

[22] R. Courant & D. Hilbert, Methods of Mathematical Physics, volume II. Interscience Publishers, 1962.

[23] U. Dierkes & S. Hildebrandt & F. Sauvigny, Minimal surfaces. Springer-Verlag, 2010.

[24] R.K. Dodd & J.C. Eilbeck & J.D. Gibbon & H.C. Morris, Solitons and nonlinear wave equations. Academic Press, London, 1982.

[25] K. Ecker, On mean curvature flow of spacelike hypersurfaces in asymptotically flat spacetimes. J. Austral. Math. Soc. Ser. A 55 (1993) 41-59.

[26] K. Ecker, Interior estimates and longtime solutions for mean curvature flow of non compact spacelike hypersurfaces in Minkowski space. J. Differential Geometry 45 (1997) 481-498.

[27] K. Ecker & G. Huisken, Parabolic methods for the construction of spacelike slices of prescribed mean curvature in cosmological spacetimes. Comm. Math. Phys. 135 (1991) 595-613.

[28] Paul Germain, La théorie générale des mouvements coniques et ses applications à l'aérodynamique supersonique. Office National d'Etudes et de Recherches Aéronautiques, publication no. 34, 1949.

[29] Paul Germain, La théorie des mouvements homogènes et son application au calcul de certain ailes delta en régime supersonique. Recherche Aéronautique 7 (1949) 3-16.

[30] C.H. Gu, The extremal surfaces in the 3-dimensional Minkowski space. Acta Math. Sinica (N.S.) 1 (1985), no. 2, 173-180.

[31] C.H. Gu, A class of boundary problems for estremal surfaces of mixed type in Minkowski 3-space. J. Reine Angew. Math. 385 (1988) 195-202.

[32] E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$. Comm. Pure Appl. Math. 3 (1950) 201-230.

[33] V.A. Klyachin, Investigation of solutions of an equation of surfaces with zero mean curvature of mixed type in Minkowski space. Dokl. Akad. Nauk 384 (2002) 587-589.

[34] O. Kobayashi, Maximal surfaces in the 3-dimensional Minkowski space. Tokyo J. Math. 6 (1983) 297-309.

[35] J.D. Lawrence, A catalog of special plane curves. Dover Publications, 1972.

[36] J. Li, Stationary surfaces in Minkowski space. I. A representation formula. Pacific J. Math. 158 (1993) 353-363.

[37] H.W. Liepmann & A. Roshko, Elements of Gasdynamics. Dover Publications, 1957 and 2001.

[38] H. Lindblad, A remark on global existence for small initial data of the minimal surface equation in Minkowskian spacetime. Proc. Amer. Math. Soc. 321 (4) (2003) 1095-1102.

[39] H.L. Liu, Minimal time-like surfaces in three-dimensional Minkowski space. J. Northeast Univ. Tech. 12 (1991) 308-310.

[40] H.L. Liu, The general Weierstrass formula for surfaces in 3-dimensional Minkowski space. Acta Math. Sinica 38 (1995) 191-199.

[41] L. Mazet, A uniqueness result for maximal surfaces in Minkowski 3-space. C. R. Acad. Sci. Paris Ser. I 344 (2007) 785-790.

[42] J.C. Nitsche, Lectures on minimal surfaces. Cambridge Univ. Press, 1989.

[43] R. Osserman, A survey of minimal surfaces. Dover Publications, 2002.

[44] C. Rogers & W.K. Schief, Bäcklund and Darboux transformations. Cambridge University Press, 2002.

[45] C. Rogers & W.K. Schief & M.E. Johnston, Bäcklund and his work: applications in soliton theory. Pages 16-55 in Geometric approaches to differential equations, P.J. Vassiliou & I.G. Lisle Editors, Cambridge University Press, 2000.

[46] C. Rogers & W.F. Shadwick, Bäcklund transformations and their applications. Academic Press (1982).

[47] A. Romero, Simple proof of Calabi-Bernstein theorem on maximal surfaces. Proc. Amer. Math. Soc. 124 (1996) 1315-1317.

[48] J. Serrin, Mathematical principles of classical fluid mechanics. Handbuch der Physik, volume 8 (1959), pages 125-263.

[49] M. Umehara & K. Yamada, Maximal surfaces with singularities in Minkowski space. Hokkaido Math. J. 35 (2006) 13-40.

[50] I. Van de Woesijne, Minimal surfaces of the 3-dimensional Minkowski space. Geometry and topology of submanifolds II (Avignon, 1988), 344-369, World Sci. Publ., 1990.

[51] D. Zwillinger, Handbook of differential equations. Academic Press, 1997.