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Recent Applications of Near Operators Campanato's Theory to Study Complex Problems[†]

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Summary

This paper contains a short survey of the last applications of Campanato's near operators theory. In particular we give an example of its applications to the proof of existence of a solution of the Cauchy-Dirichlet problem concerning a class of fully nonlinear parabolic systems.

Key words: *Campanato's near operators, fully nonlinear parabolic systems, MEMS, Von Kármán equations.*

Riassunto

Questo articolo contiene una breve panoramica delle ultime applicazioni della teoria degli operatori vicini di Campanato. In particolare delle sue applicazioni alla dimostrazione dell'esistenza di soluzione del problema di Cauchy -Dirichlet riguardante una classe di sistemi parabolici totalmente non lineari.

Parole chiave: *Teoria degli operatori vicini di Campanato, sistemi parabolici totalmente non lineari, MEMS, equazioni di Von Kármán.*

1 Introduction

In this paper we want to explain the main results and the strategy that make possible to solve some problems concerning partial differential equations or systems by *Campanato' s near operators theory*.

This theory was created by Sergio Campanato at the end of the years '80 in the last century, to show the existence of solutions of nonvariational elliptic equations with bounded and measurable coefficents verifying the so called Cordes Condition (see for example S. Campanato [2]). The first result obtained was a surijectivity theorem for operators between Hilbert spaces. Afterwards this theory was extended to obtaining new results and new ambits of applications. Even after

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some ten-year period from its introduction, we can certainly assert that the simplicity is its main characteristic property. Indeed S. Campanato used to say kindly during his lectures on this matter "....I like *the simple maths*, that you can relate to Crispi street news-seller...".

In this paper we will expose some results obtained by this: **simple math in the Campanato's meaning**. Obviously "simple" don't mean trivial. Indeed the depth and the importance of some results of this theory make us to be able to solve, as we will see in the following sections, very complex problems.

In our opinion we can expand to it the judgement that Enrico Giusti (see E. Giusti [12]) passed for the other Campanato's theory, that one of elliptic regularity:

.... The method developed by Campanato had a period of success in the years between 1965 and 1968, but since then it seems to be gone back in the shadows, up to the point of not finding place in the books on this subject; this is wrong in my opinion, because in addition to its **peculiarities of good style***, its potentials seem still today far from being exhausted.*

In the Section 2 we outline briefly the main results of the near operator theory. In the Sections 3, 4, 5 we give an example of its applications to a Cauchy-Dirichlet Problem for a fully nonlinear parabolic system. In the section 6 and 7 we expose its applications to some other problems.

2 A short survey of near operators theory

Definition 2.1. *Let $\mathcal{B}$ be a Banach space with norm $\|\cdot\|$ and $\mathcal{X}$ be a set. Let A, B be two operators such that $A, B : \mathcal{X} \longrightarrow \mathcal{B}$. We say that A is* near *B if there are two positive constants $\overline{a}$, k, with $0 < k < 1$, such that for any $x_1,\ x_2 \in \mathcal{X}$ we have:*

$$\|B(x_1) - B(x_2) - \overline{a}\,[A(x_1) - A(x_2)]\| \leq k\|B(x_1) - B(x_2)\|.$$

We underline that the operators are defined in a set, than we haven't conditions which deal at structure of their domain. Moreover to say that the operator A is near operator B means to give a control of the oscillations of A by means of those ones of B.

If $\mathcal{X}$ is a Banach space and A is Frechét differentiable, the differential dA of A plays the rule of B.

Generally, as we will see, B acts as a differential with all that it follows.

We also remark that, in particular,if $\mathcal{X}$ is a Banach space and A is Frechét differentiable, in $x_0 \in \mathcal{X}$ and its differentiable $dA(x_0)$ is invertible in this point, then A is near $dA(x_0)$ in a neighborhood of x_0 (see A. Tarsia [16]).

The following theorem explain the properties that keep by nearness (see S. Campanato [4], A. Tarsia [14], A. Tarsia [15]).

Theorem 2.1. *Let A be near to B, then :*

(i) if B is injective (surjective) then A is injective (surjective) too;

(ii) if $B(\mathcal{X})$ is open in $\mathcal{B}$ then $A(\mathcal{X})$ is open in $\mathcal{B}$;

(iii) if $B(\mathcal{X})$ is dense in $\mathcal{B}$ then $A(\mathcal{X})$ is dense in $\mathcal{B}$;

(iv) if $B(\mathcal{X})$ is compact in $\mathcal{B}$ then $A(\mathcal{X})$ is compact $\mathcal{B}$.

The proposition (ii) in the case of an operator which has a differential that is invertible coincides exactly with the local invertibility theorem and this agrees with the fact that B plays the rule of differential.

In conclusion the theorem asserts that if A is near an operator B which is a "good" one then also A is "good".

This suggest the strategy that allows to solve some differential problems by this theory: to show that A has a certain property we must find an operator B, that has this property and such that A is near B.

What we have observed, it seems confirm the comments presented in the introduction regarding the simplicity of near operators theory. But we underline that "simple" do not mean trivial. Indeed to apply this theory we must pass various difficulties, in particular, one of this is to find an operator B which is "good" and to show that A is near B. Generally all this needs inequalities rather elaborated.

Obviously the complexity of the problem that we deal imply that it is impossible to proceed trivially.

Moreover we observe that the Proposition (ii) of Theorem 2.1 is an open mapping theorem.

We remark that for every open mapping theorem there exists consequentially an Implicit Function Theorem. Also in this case it had been shown (see A. Tarsia [16]) the following:

Theorem 2.2. *(Implicit Functions Theorem). Let X be a topological space, Y a set, Z a Banach space, $\mathcal{F} : X \times Y \longrightarrow Z$, $B : Y \longrightarrow Z$. Suppose that*

$$(x_0, y_0) \in X \times Y \text{ exists such that } \mathcal{F}(x_0, y_0) = 0; \tag{1}$$

$$\text{the map } x \to \mathcal{F}(x, y_0) \text{ is continuous in } x = x_0; \tag{2}$$

there exist positive numbers $\tilde{\alpha}, k$, with $k \in (0, 1)$, and a neighbourhood $U(x_0) \subset X$ of x_0, such that for any y_1, $y_2 \in Y$, and for any $x \in U(x_0)$

$$\|B(y_1) - B(y_2) - \tilde{\alpha}[\mathcal{F}(x, y_1) - \mathcal{F}(x, y_2)]\|_Z \leq k\|B(y_1) - B(y_2)\|_Z \tag{3}$$

$$B \text{ is injective ;} \tag{4}$$

$$B(Y) \text{ is a neighbourhood of } z_0 = B(y_0). \tag{5}$$

Then the following is true: there exist a ball $S(z_0, \rho) \subset B(Y)$ and a neighbourhood $V(x_0) \subset U(x_0)$ of x_0, such that there is exactly one solution $y = y(x) : V(x_0) \longrightarrow B^{-1}(S(z_0, \rho))$ of the following problem:

$$\begin{cases} \mathcal{F}(x, y(x)) = 0, \text{ for any } x \in V(x_0), \\ \\ y(x_0) = y_0. \end{cases} \tag{6}$$

We underline that the statement of this theorem is similar to the others Implicit Function Theorems, but in this one the assumption of differentiability of the function is replaced by (3), that is a nearness assmption.

Moreover we obtain a further result, consequence of this theory, proving the following "nonlinear continuity method" (see L. Fattorusso, A. Tarsia [9]). This is a generalization of the well-known method of continuity for linear operators as well as for nonlinear operators (see for example D. Gilbarg, N. S. Trudinger [11]), because we do not assume the hypothesis that the operator is differentiable.

Theorem 2.3. *Let $\mathcal{B}$ be a Banach space normed with $\|\cdot\|$ and let $\mathcal{X}$ be a set. Let $\{A_r\}_{r \in I}$ be a set of operators, $I \subset \mathbb{R}$ an interval, $A_r : \mathcal{X} \longrightarrow \mathcal{B}$. If there exists a positive constant c such that for any s, $r \in I$ and $u, v \in \mathcal{X}$ we have*

$$\|A_r(u) - A_r(v) - [A_s(u) - A_s(v)]\| \leq c\,|r - s|\,\|A_r(u) - A_r(v)\|, \tag{7}$$

then

1. *if A_r is injective (surjective) for some $r \in I$ then A_r is injective (surjective) for any $r \in I$.*
2. *if $A_r(\mathcal{X})$ is open (closed) for some $r \in I$ then $A_r(\mathcal{X})$ is open (closed) for any $r \in I$.*
3. *if $A_r(\mathcal{X})$ is dense for some $r \in I$ then $A_r(\mathcal{X})$ is dense for any $r \in I$.*

3 An example of application to a fully nonlinear parabolic problem

In this section we expose the strategy that we follows appling the near operators theory to a Cauchy-Dirichlet problem for a fully nonlinear parabolic system.

This result, that we here expounde briefly, is contained in L. Fattorusso, A. Tarsia [10].

Let Ω be an convex open bouned set in $\mathbb{R}^n$, $n \geq 2$, with the boundary $\partial\Omega$ of $C^{2,1}$ class, N a positive integer. Let $u: \ \Omega \times [0,T] \longrightarrow \mathbb{R}^N$ be a function, $T > 0$.

Let us consider the Cauchy-Dirichlet problem

$$\begin{cases} u \in L^2(0,T,H^2 \cap H_0^1(\Omega,\mathbb{R}^N)) \cap C^0([0,T],H_0^1(\Omega,\mathbb{R}^N)) \cap H^1(0,T,L^2(\Omega,\mathbb{R}^N)). \\ \\ \mathbf{a}(x,t,u(x,t),Du(x,t),D^2u(x,t)) \ - \ u'(x,t) = 0, \ \text{ a.e. in } \Omega \times [0,T], \\ \\ u(x,0) \ = \ 0 \text{ a.e. in } \Omega, \end{cases}$$

where $\mathbf{a} : \Omega \times [0,T] \times \mathbb{R}^N \times \mathbb{R}^{nN} \times \mathbb{R}^{n^2N} \longrightarrow \mathbb{R}^N$ and $\mathbf{a}(x,t,0,0,0) \in L^2(\Omega \times [0,T],\mathbb{R}^N)$. Our purpose is to show that, under suitable assumptions on $\mathbf{a}$, this Problem is well posed. In order to make this, there is no loss of generality in writing the problem as follows:

$$\begin{cases} u \in L^2(0,T,H^2 \cap H_0^1(\Omega,\mathbb{R}^N)) \cap C^0([0,T],H_0^1(\Omega,\mathbb{R}^N)) \cap H^1(0,T,L^2(\Omega,\mathbb{R}^N)). \\ \\ F(x,t,u(x,t),Du(x,t),D^2u(x,t)) - u'(x,t) = \\ \\ = g(x,t,u(x,t),Du(x,t)) + f(x,t), \ \text{ a.e. in } \Omega \times [0,T], \\ \\ u(x,0) \ = \ 0 \text{ a.e. in } \Omega, \end{cases} \tag{8}$$

where $F(x,t,u,Du,0) = 0$ and $g(x,t,0,0) = 0$([1]).

We show the global existence of solutions of the problem without differentiability hypothesis on F respect to the variable D^2u. We can obtain this result assuming on F the so called ellipticity hypothesis of Campanato, that before, in his papers, he called "Nonlinear Cordes Condition" (see, for example, S. Campanato [2]), and afterwards "Condition A" (see S. Campanato [3]), that is the following:

Definition 3.1. (**Condition A** *or* **Campanato condition of ellipticity**)

The operator $F : \Omega \times [0,T] \times \mathbb{R}^N \times \mathbb{R}^{nN} \times \mathbb{R}^{n^2N} \longrightarrow \mathbb{R}^N$ verifies Condition A if there exist tree real constants γ, δ, a with $\gamma, \delta \geq 0$, $\gamma + \delta < 1$, $a > 0$ such that for any $u \in \mathbb{R}^n$, $p \in \mathbb{R}^{nN}$, $\mathcal{M}, Q \in \mathcal{S}_N^n$ ([2]) and for a. e. $(x,t) \in \Omega \times [0,T]$ we have

$$\begin{gathered} \left\| \sum_{i=1}^n Q_{ii} \ - \ a\,[F(x,t,u,p,\mathcal{M}+Q) \ - \ F(x,t,u,p,\mathcal{M})] \right\|_N \ \leq \\ \leq \gamma\, \|Q\|_{n^2N} \ + \ \delta \ \left\| \sum_{i=1}^n Q_{ii} \right\|_N . \end{gathered} \tag{10}$$

[1] Indeed it is enough to assume

$$\begin{aligned} F(x,t,u(x,t),Du(x,t),D^2u(x,t)) &= \mathbf{a}(x,t,u(x,t),Du(x,t),D^2u(x,t)) - \mathbf{a}(x,t,u(x,t),Du(x,t),0), \\ g(x,t,u(x,t),Du(x,t)) &= -[\mathbf{a}(x,t,u(x,t),Du(x,t),0) - \mathbf{a}(x,t,0,0,0)], \\ f(x,t) &= -\mathbf{a}(x,t,0,0,0). \end{aligned} \tag{9}$$

[2] $p = (p^1, \cdots, p^n)$, $p^i \in \mathbb{R}^N$, if $p \in \mathbb{R}^{nN}$. $(\ |\ )_N$ and $\|\ \|_N$ are respectively the scalar product and the norm in $\mathbb{R}^N$. $\mathcal{S}_N^n$ is the vector space of N-ples $Q = (\{Q_{ij}^1\}_{i,j=1,\cdots,n}, \cdots, \{Q_{ij}^N\}_{i,j=1,\cdots,n})$ of $n \times n$ matrices with $Q_{ij}^k = Q_{ji}^k$, $i, j = 1, \cdots, n$, $k = 1, \cdots, N$, equipped with the scalar product: $(\mathcal{M}|Q)_{n^2N} = \sum_{i,j=1}^n (\mathcal{M}_{ij}|Q_{ij})_N$.

It is possible to show that the Condition A_x, i. e. Condition A in which the constant a is replaced by a positive, bounded function, is equivalent to the Cordes Conditon, if the operator is linear, scalar with L^∞ coefficents (see A. Tarsia [17]).

Moreover we can confront Condition A with others ellipticity conditions (see A. Tarsia [18]).

The other assumptios that we put to solve the problem are of two tipes:

assumptions on the growth in u and p and "compatibility assumptions" between the terms containing second order derivatives and those containing lower order derivatives.

In particular we consider the following assumptions:

$$F \text{ is measurable in } (x,t) \text{ and continuous in the other variables;} \tag{11}$$

$$\begin{aligned}&\text{there exists a function } \omega : \mathbb{R}^N \times \mathbb{R}^{nN} \longrightarrow \mathbb{R}^+ \text{ bounded and continuous in } (u,p),\\ &\text{with } \omega(0,0)=0 \text{ and such that for any } u_1,\ u_2 \in \mathbb{R}^N,\ p_1,\ p_2 \in \mathbb{R}^{nN},\ \xi \in \mathbb{R}^{n^2N},\\ &\|F(x,t,u_1,p_1,\xi) - F(x,t,u_2,p_2,\xi)\|_N \le \omega(u_1-u_2, p_1-p_2)\|\xi\|_{n^2N},\ \text{ a.e. in } \Omega\times[0,T];\end{aligned} \tag{12}$$

$$\begin{aligned}&\text{there exist three positive constants} M,\ \alpha,\ \beta \text{ such that for any } u\in\mathbb{R}^N,\ p\in R^{nN} \text{ it results}\\ &\|g(x,t,u,p)\|_N \le M(\|u\|_N^\alpha + \|p\|_{nN}^\beta),\ \text{ a.e. in } \Omega\times[0,T],\ \text{ where}\\ &\text{if} \quad n=2, \quad \alpha\ge 1, \qquad\qquad \beta=1,\\ &\text{if} \quad 2<n, \quad 1\le\alpha\le\frac{n}{n-2}, \quad \beta=1.\end{aligned} \tag{13}$$

$$\begin{aligned}&g \text{ is a differentiable function in the variables } (u,p) \text{ and we assume that there exist}\\ &M_1,\ M_2>0, \text{ such that for any } u\in\mathbb{R}^N,\ p\in\mathbb{R}^{nN} \text{ it results}\\ &\left\|\frac{\partial g(x,t,u,p)}{\partial u}\right\|_{N^2} \le M_1\left(\|u\|_N^{\alpha-1} + \|p\|_{nN}\right),\ \text{ a. e. in } \Omega\times[0,T];\\ &\left\|\frac{\partial g(x,t,u,p)}{\partial p}\right\|_{nN^2} \le M_2\ \|u\|_N^\alpha,\ \text{ a. e. in } \Omega\times[0,T].\end{aligned} \tag{14}$$

Where $\frac{\partial g(x,t,u,p)}{\partial u} = \{\frac{\partial g_i(x,t,u,p)}{\partial u_j}\}_{i,j=1,\cdots,N}$ and $\frac{\partial g(x,t,u,p)}{\partial p} = \{\frac{\partial g_i(x,t,u,p)}{\partial p_j^k}\}_{i,j=1,\cdots,N;\,k=1,\cdots,n}$.

Moreover we suppose that at least one of the following conditions is verified:

$$\begin{aligned}&\text{for any } u,\ v\in H \text{ it results}\\ &\int_0^T\int_\Omega \Big(F(x,t,u,Du,D^2u) - F(x,t,v,Dv,D^2v)+\\ &-[u'(x,t)-v'(x,t)]|g(x,t,u,Du)-g(x,t,v,Dv))_N \quad dx\,dt \le 0;\end{aligned} \tag{15}$$

for any $u,\ v\in H$ it results

$$\int_0^T\int_\Omega (\Delta u - \Delta v - a[u'(x,t)-v'(x,t)]|g(x,t,u,Du)-g(x,t,v,Dv))_N \quad dx\,dt \le 0. \tag{16}$$

Without at least one of the assumptions (15) and (16) we can't state that the problem has solution ([3]), while to have uniquenness of solution we will assume that in (15) the inequality is stricly.

More precisely we can show the following:

Theorem 3.1. *We assume that the Condition A holds , and the hypotheses (11), (12), (13), (15) or (16) are verified. If $f \in L^2(\Omega \times [0,T], \mathbb{R}^N)$ then the Cauchy-Dirichlet Problem (8) has at least one solution, that it is unique if (15) holds with a strict inequality.*

To prove this result we procede into a sequence of steps. In the first step (see Section 3) we prove the existence and the uniqueness of solution for the Cauchy-Dirichlet problem, in the case of the following systems:

$$F(x,t,D^2u(x,t)) - u'(x,t) = g(x,t,u(x,t),Du(x,t)) + f(x,t).$$

We make this by using a particolar Implicit Functions theorem, deriving by "near operators "theory (see Theorem 2.2), for which differentiability hypothesis on the function is not necessary, and a generalization of the continuity method (see Theorem 2.3).

In the second step we prove the existence in the case of the systems written in complete form (see Section 5), making use of the Fixed point Theorem of Schauder-Tychonov.

4 The system with principal part $F(x,t,D^2u)$

Let us consider the following Cauchy-Dirichlet problem:

$$\begin{cases} u \in L^2(0,T,H^2 \cap H_0^1(\Omega,\mathbb{R}^N)) \cap C^0([0,T],H_0^1(\Omega,\mathbb{R}^N)) \cap H^1(0,T,L^2(\Omega,\mathbb{R}^N)), \\ F(x,D^2u(x,t)) - u'(x,t) = g(x,u(x,t),Du(x,t)) + f(x,t), \text{ a. e. in } \Omega \times [0,T], \\ u(x,0) = 0, \text{ a. e. in } \Omega. \end{cases} \tag{17}$$

To show the existence and the uniqueness of the solution of this problem we will use the above mentioned theorem of nonlinear continuity (Theorem 2.3). In order to make this we apply the Implicit Function Theorem (Theorem 2.2) to the following problem:

$$\begin{cases} u \in L^2(0,T,H^2 \cap H_0^1(\Omega,\mathbb{R}^N)) \cap C^0([0,T],H_0^1(\Omega,\mathbb{R}^N)) \cap H^1(0,T,L^2(\Omega,\mathbb{R}^N)), \\ F(x,t,D^2u(x,t)) - u'(x,t) = r\,g(x,t,u,Du) + f(x,t), \text{ a. e. in } \Omega \times [0,T], \\ u(x,0) = 0 \text{ a. e. in } \Omega, \end{cases} \tag{18}$$

in which the nonlinear therms that contains the first derivatives appears with a penality parameters. Therefore now we show:

Theorem 4.1. *If we assume Condition A, and hypotheses (11), (12), (13), (14), (15) or (16). If $f \in L^2(\Omega \times [0,T], \mathbb{R}^N)$, then there exists $r_0 > 0$ such that for any $r \in (-r_0, r_0)$, the problem has one and only one solution.*

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[3]We remark that these conditions are necessary in the case of linear elliptic equations since, for example, if $\lambda > 0$ is a eigenvalue of Δ, then as everybody knows, the problem

$$\begin{cases} u \in H^2 \cap H_0^1(\Omega,\mathbb{R}^N) \\ \Delta u(x) + \lambda u(x) = f(x), \text{ q.o. in } \Omega, \end{cases}$$

isn't well posed.

Proof. We use Implicit Function Theorem 2.2 with the following plan, where, for semplicity we assume

$H = L^2(0,T,H^2 \cap H_0^1(\Omega,\mathbb{R}^N)) \cap C^0([0,T],H_0^1(\Omega,\mathbb{R}^N)) \cap H^1(0,T,L^2(\Omega,\mathbb{R}^N))$:

$X = \mathbb{R}$; $Z = L^2(\Omega \times [0,T],\mathbb{R}^N)$;

$Y = \{u : \ u \in H, \text{ with } \|u - u_0\|_H < R\}$, where $R > 0$ is fixed and we determine it in successive proof.

$\mathcal{F}(s,u) = a\,F(x,t,D^2u) - \ a\,u'(x,t) - s\,a\,g(x,t,u,Du) - a\,f(x,t)$, $s \in \mathbb{R}$;

$(s_0,u_0) = (0,u_0)$ where $u_0 \in H$ is a solution of system

$F(x,t,D^2u_0(x,t)) - u_0'(x,t) = f(x,t)$, a. e. in $\Omega \times [0,T]$,

so that $\mathcal{F}(s_0,u_0) = 0$;

$B(u) = a\,F(x,t,D^2u) - \ a\,u'(x,t)$.

We observe that

$\mathcal{F} : \ X \times Y \longrightarrow Z$, because $s \in \mathbb{R}$, $u \in H$; moreover we have

$$\|\mathcal{F}(s,u)\|_Z^2 = \int_0^T \int_\Omega \|a\,F(x,t,D^2u) - s\,a\,g(x,t,u,Du) - a\,f(x,t)\|_N^2\,dx\,dt \leq$$
$$\leq c\,a^2 \int_0^T \int_\Omega [\|F(x,t,D^2u)\|_N^2 + |s|^2\|g(x,t,u,Du)\|_N^2 + \|f(x,t)\|_N^2]\ dx\,dt \ < \ +\infty,$$

by assumption on f e g, by Sobolev imbedding theorems, we have for any $u \in H$ and $\alpha \leq \frac{n}{n-2}$ if $n > 2$, or $\alpha \geq 1$ if $n = 2$, and $\beta = 1$:

$$\int_0^T \int_\Omega \|g(x,t,u,Du)\|_N^2 dx\,dt \ \leq \ c \int_0^T \int_\Omega \|u\|_N^{2\alpha}\,dx\,dt \ + \ c \int_0^T \int_\Omega \|Du\|_{nN}^{2\beta}\,dx\,dt \ < +\infty.$$

Moreover by Condition A we obtain

$$\int_0^T \int_\Omega \|a\,F(x,t,D^2u)\|_N^2\ dx\,dt \leq \ c \int_0^T \int_\Omega \Big(\|a\,F(x,t,D^2u(x,t)) - \Delta u(x,t)\|_N^2 +$$
$$+\|\Delta u(x,t)\|_N^2\Big)\ dx\,dt \leq$$
$$\leq \frac{2}{\sigma^2} \int_0^T \int_\Omega \Big(\gamma\|D^2u(x,t)\|_{n^2N}^2 + (\delta+1)\|\Delta u(x,t)\|_N^2\Big)\ dx\,dt < +\infty.$$

The last inequality show that

$B : H \longrightarrow L^2(\Omega \times [0,T],\mathbb{R}^N)$.

Moreover as consequnce of Condition A, we prove that B is near $\Delta - \ a\,\frac{d}{dt}$ as operator between Y and Z.

By Theorem 2.1 we can assert that, since $\Delta - \ a\,\frac{d}{dt}$ is a bijection between H and $L^2(\Omega \times [0,T],\mathbb{R}^N)$, it results that B also is a bijection between H and $L^2(\Omega \times [0,T],\mathbb{R}^N)$. On the other hand $\Delta - \ a\,\frac{d}{dt}$ is an open map, by Banach open map Theorem, then B also is an open map. In particular $B(Y)$ is open in Z. Moreover the function $s \longrightarrow \mathcal{F}(s,u)$ is continuous in $s = 0$. The proof is completed by showing that the assumption (3) of Theorem 2.2 is verified:

there exists $k \in (0,1)$ and $r_1 > 0$ such that for any $u_1, u_2 \in Y$ and $s \in (-r_1,r_1)$ we have

$$\int_0^T \int_\Omega \|a\, F(x,t,D^2u_1) - a\, u_1'(x,t) - [a\, F(x,t,D^2u_2) - a\, u_2'(x,t)] +$$
$$- \Big\{ a\, F(x, D^2u_1(x)) - \ a\, u_1'(x,t) - s\, g(x,t,u_1,Du_1)] +$$
$$-[a\, F(x,t,D^2u_2) - \ a\, u_2'(x,t) - s\, g(x,t,u_2,Du_2)] \Big\} \|_N^2 \ dx\, dt \le$$
$$\le k\, a^2 \int_0^T \int_\Omega \|\, F(x,t,D^2u_1) - u_1'(x,t) - [\, F(x,t,D^2u_2) - u_2'(x,t)\|_N^2 \ dx\, dt.$$

We obtain these inequality after very long and laborious calculations, that for to sake brevity, we don't insert, assuming $k = s^2\, c(d_\Omega, n, \gamma, \delta)(R + \|u_0\|_H)^\xi$, because u_1, $u_2 \in Y$ implies $\|u_1\|_H \le \|u_0\|_H + R$, $\|u_2\|_H \le \|u_0\|_H + R$. Then we have (3), accordingly $|s| \in (0, r_1)$ with r_1 such that $r_1^2\, c(d_\Omega, n, \gamma, \delta)(R + \|u_0\|_H)^\xi < 1$. So $(-r_1, r_1)$ is the neighbourhood $U(s_0)$ of $s_0 = 0$ on which is true inequality (3). In this way we have shown that all assumptions of Theorem 2.2 are verified. By this Theorem, we can conclude that there exists a ball $S(F(\cdot, D^2u_0), \rho))$ in $B(Y)$ and $r_0 \in (0, r_1)$, such that there exists one and only one solution u_s, with $s \in (0, r_0)$, $u_0 \in B^{-1}(S(F(\cdot, D^2u_0), \rho))$, solution of([4])

$$\begin{cases} \mathcal{F}(s, u_s) = 0 \\ u_{s|s=0} = u_0, \end{cases}$$

that is there is one and only one solution in the neighbourhood before determined that is solution of the problem

$\mathcal{F}(s,u) = a\, F(x,t,D^2u) - \ a\, u'(x,t) - s\, a\, g(x,t,u,Du) - a\, f(x,t) = 0.$ □

Now we consider the Cauchy-Dirichlet problem with the penalization parameter $s = 1$ and we prove the following theorem

Theorem 4.2. *Let the Condition A, (11), (12), (13), (14), (15) or (16) be satisfied. Then if $f \in L^2(\Omega \times [0,T], \mathbb{R}^N)$ problem 17 has one and only one solution.*

We deduce the thesis applying the nonlinear method of continuity showing that there exists a positive constant c such that for any s, $r \in [r_0, 1]$ (where r_0 is estabilished by above theorem) and for any u, $v \in H$ we have

$$\|A_r(u) - A_r(v) - [A_s(u) - A_s(v)]\|_{L^2(\Omega\times[0,T],\mathbb{R}^N)} \le c\,|r-s| \|A_r(u) - A_r(v)\|_{L^2(\Omega\times[0,T],\mathbb{R}^N)},$$

where

$$A_r(u) = F(x,t,D^2u(x,t)) - u'(x,t) - r\, g(x,t,u(x,t),Du(x,t)).$$

That is we have to show following inequality

$$\int_0^T \int_\Omega \|F(x,t,D^2u) - u' - rg(x,t,u,Du) - [F(x,t,D^2v) - v' - rg(x,t,v,Dv)] +$$
$$-\{F(x,t,D^2u) - u' - sg(x,t,u,Du) - [F(x,t,D^2v) - v' - sg(x,t,v,Dv)]\}\|_N^2 \ dx\, dt \le$$
$$\le C_1|r-s|^2 \int_0^T \int_\Omega \|F(x,t,D^2u) - u' - rg(x,t,u,Du) +$$
$$-[F(x,t,D^2v) - v' - rg(x,t,v,Dv)]\|_N^2 dx\, dt.$$

[4] where $r_0 \le r_1$, so that $(-r_0, r_0)$ is the neighbourhood $V(x_0)$ of Theorem 2.2.

For make this, we show that, taking $r \in [r_0, 1]$ and $u, v \in H$, we have

$$\int_0^T \int_\Omega \|g(x,t,u,Du) - g(x,t,v,Dv)\|_N^2 dx\,dt \leq$$

$$\leq C_1 \int_0^T \int_\Omega \|F(x,t,D^2u) - u' - [F(x,t,D^2v) - v'] - r\,[g(x,t,u,Du) - g(x,t,v,Dv)]\|_N^2 dx\,dt.$$

Indeed, for any $u, v \in H$ and $r > 0$ it holds

$$\int_0^T \int_\Omega \|g(x,t,u,Du) - g(x,t,v,Dv)]\|_N^2 dx\,dt +$$

$$+\frac{1}{r}\int_0^T \int_\Omega \|F(x,t,D^2u) - u' - [F(x,t,D^2v) - v']\|_N^2\,dx\,dt \leq$$

$$\leq \frac{1}{r}\int_0^T \int_\Omega \|F(x,t,D^2u) - u' - [F(x,t,D^2v) - v'] - r\,[g(x,t,u,Du) - g(x,t,v,Dv)]\|_N^2 dx\,dt.$$

That is, for any $r \geq r_0$

$$\begin{aligned} \int_0^T \int_\Omega \|g(x,t,u,Du) - g(x,t,v,Dv)]\|_N^2 dx\,dt \ \leq \\ \leq \frac{1}{r}\ \|A_r(u) - A_r(v)\|_{L^2(\Omega\times[0,T],\mathbb{R}^N)} \leq \ \frac{1}{r_0}\ \|A_r(u) - A_r(v)\|_{L^2(\Omega\times[0,T],\mathbb{R}^N)}. \end{aligned} \tag{19}$$

This lastone inequality concludes the proof and, by untuition, we know why we can't apply as first step the nonlinear continuity method. Indeed we can't apply this one in the interval $[0,1]$ because inequality (19) is not verified in this interval, but it holds in the interval $[r_0, 1]$, with $r_0 > 0$.

5 Existence and uniquenness of solution for the system in complete form

Let us consider the Dirichlet Problem in its complete form. We prove the following theorem.

Theorem 5.1. *Let F, g satisfy respectively Condition A, hypotheses (11), (12), (13), (14), (15). If $f \in L^2(\Omega \times [0,T], \mathbb{R}^N)$, the problem (8) has one solution, that is unique if (15) holds with strictly inequality.*

To show the thesis we use the result of the above section concerning the problem with the parameter of penalization and we show the existence of a solution of the system

$$F(x,t,u(x,t),Du(x,t),D^2u(x,t)) - u'(x,t) = s\,g(x,t,u(x,t),Du(x,t)) + f(x,t),$$

where s is small enough, and then we apply the nonlinear method of contiuity.

To make this we first show following Lemma.

Lemma 5.1. *Let the hypotheses of Theorem 5.1 holds, then there exists $r_0 > 0$ such that for each $s \in [-r_0, r_0]$ the system has one solution in $L^2(0,T,H^2 \cap H_0^1(\Omega,\mathbb{R}^N)) \cap C^0([0,T],H_0^1(\Omega,\mathbb{R}^N)) \cap H^1(0,T,L^2(\Omega,\mathbb{R}^N))$.*

In order to apply the Schauder-Tychonov theorem we consider

$$\mathcal{T} : L^2(0,T,H_0^1(\Omega,\mathbb{R}^N)) \longrightarrow L^2(0,T,H^2 \cap H_0^1(\Omega,\mathbb{R}^N)) \cap C^0([0,T],H_0^1(\Omega,\mathbb{R}^N)) \cap H^1(0,T,L^2(\Omega,\mathbb{R}^N)),$$

such that maps any $w \in L^2(0,T,H^1_0(\Omega,\mathbb{R}^N))$ into $u = \mathcal{T}(w)$ solution of problem

$$\begin{cases} u \in L^2(0,T,H^2 \cap H^1_0(\Omega,\mathbb{R}^N)) \cap C^0([0,T],H^1_0(\Omega,\mathbb{R}^N)) \cap H^1(0,T,L^2(\Omega,\mathbb{R}^N)), \\ F(x,t,w(x,t),Dw(x,t),D^2u(x,t)) - u'(x,t) = \\ = s\,g(x,t,u(x,t),Du(x,t)) + f(x,t), \text{ q. o. in } \Omega \times [0,T], \end{cases} \tag{20}$$

The principal part does not depend on u and Du so, by Theorem 4.1, if s is small enough, we know that the problem has one and only one solution.

Now we can consider the following imbeddings

$J_1 : L^2(0,T,H^2 \cap H^1_0(\Omega,\mathbb{R}^N)) \cap H^1(0,T,L^2(\Omega,\mathbb{R}^N)) \longrightarrow H^\theta(0,T,H^{2(1-\theta)}(\Omega,\mathbb{R}^N)), \quad \theta \in [0,1]$

$J_2 : H^\theta(0,T,H^{2(1-\theta)}(\Omega,\mathbb{R}^N)) \longrightarrow L^2(0,T,H^1_0(\Omega,\mathbb{R}^N)), \theta \in (0,\frac{1}{2})$.

J_1 is a continuous map, while J_2 is a compct map by Rellich imbedding theorem. We can set

$$\mathcal{T}_1 = J_2 \circ J_1 \circ \mathcal{T}.$$

Now the thesis follows by applying Schauder-Tychonov theorem to the map $\mathcal{T}_1$. To make this we show by laborious calculations, that for the sake of brevity, we don't insert, that:

- the set of solutions of (20), for any $w \in L^2(0,T,H^1_0(\Omega,\mathbb{R}^N))$, is bounded;
- $\mathcal{T}$ is continuous.

At last to show the existence of the solution of the problem with $r = 1$, we apply to the set of operators $A_r(u) = F(x,t,u,Du,D^2u) - u' - s\,g(x,t,u,Du)$ the nonlinear method of continuity, proceeding with inequality of the same type of these used in the previous section assuming $\mathcal{X} = H$, $\mathcal{B} = L^2(\Omega \times [0,T],\mathbb{R}^N)$ so as to have $A_t : \mathcal{X} \longrightarrow \mathcal{B}$.

6 Some others results obtained by near operators theory

The near operators theory of Campanato in the begining, as we underlined in the introduction, have been made to solve the second order elliptic problems. Recently these methods have been applyed to higher order (fourth order) problems, that is, to problems which present their principal part that consists of biharmonic operators. In this section we give some examples.

(a) Von Kármán Equations.

To calculate the in-plane stresses in a thin plate it is necessary to solve a biharmonic equation involving the Airy stress function. At the same time, to calculate the deflection of the plate we need to solve a biharmonic equation. In 1910 Theodor von Kármán considered that these two effect can act simultaneously and then he proposed a system of two equations in the stress function and in the transverse displacement (see T. Von Kármán [19]).

Many authors have studied this problem: a complete study and a very rich bibliography can be found on a recent book by I. Chueshov and I. Lasieka [8]. We have considered the case where the nonlinear terms have nonconstant coefficients. This choice describes the case where the plate is subject to a non-uniform state of plane stress. More precisely, we consider the following Dirichlet problem

$$\begin{cases} \Delta^2 w(x,y) + a_1(x,y)\,v_{yy}(x,y)\,w_{xx}(x,y) + a_2(x,y)v_{xy}(x,y)\,w_{xy}(x,y) + \\ + a_3(x,y)\,v_{xx}(x,y)\,w_{yy}(x,y) = f(x,y), & \text{on } \Omega, \\ \Delta^2 v(x,y) + b_1(x,y)w_{xx}(x,y)\,w_{yy}(x,y) + b_2(x,y)\,w^2_{xy}(x,y) = g(x,y), & \text{on } \Omega, \\ v(x,y) = w(x,y) = \frac{\partial v(x,y)}{\partial \nu} = \frac{\partial w(x,y)}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

here Ω is a bounded open set of $\mathbb{R}^2$, with boundary $\partial\Omega$ of class $C^{4,1}$, and $f \in L^{p_1}(\Omega)$, $g \in L^{p_2}(\Omega)$, $1 < p_1, p_2$. The functions a_1, a_2, a_3 are in $L^{r_1}(\Omega)$ and b_1, b_2 are in $L^{r_2}(\Omega)$, $p_1 < r_1$, $p_2 < r_2$.

We studied existence and uniqueness of the solution $(w, v) \in H^{4,p_1} \cap H^{2,p_1}(\Omega) \times H^{4,p_2} \cap H^{2,p_2}(\Omega)$ by a new technique, different from the usual ones used to study the problem with $p_1 = p_2 = 2$ and constant coefficents: $a_1 = a_2 = a_3 = b_1 = b_2 = 1$.

In our paper (see L. Fattorusso, A. Tarsia [9]) we have given, by Campanato's near operators tecniques, a new contribution to this theory, and then we exploit it to show that problem (21) is well posed in the space $H^{4,p_1} \cap H_0^{2,p_1}(\Omega) \times H^{4,p_2} \cap H_0^{2,p_2}(\Omega)$ when the data are small enough. But in many cases the data are not small, either because the diameter of $\mathbf{\Omega}$ is large as the coefficients a_1, a_2, a_3, b_1, b_2 exceed a give threshold.

Thus we prove only the existence of solutions but not their uniqueness. The inspection of simultaneous presence of the bifurcating solutions constitutes a classical branch of the theory of slender structures.

(b) MEMS Equations (Micro Electro Mechanical Systems Equations).

In an another paper (see D. Cassani, L. Fattorusso, A. Tarsia [7]) we apply the Campanoto's methods to study a time dependent nonlocal fourth order equation which is a model for describing electrostatic actuation in MEMS. devices. From the mathematical point of view, we can think of a plate problem set on a micro-scale in which usual first order approximations, acceptable in the standard "visible" scale, loose their validity and where one needs to take into account nonlocal effects which in this context are not negligible. Precisely, we consider the following problem

$$\begin{cases} \Delta^2 u + c(x,t)\,u' + u'' = G(\beta,\gamma,u) + H(\lambda(t),\chi,p(x),u), \text{ in } \Omega\times[0,T] \\ 0 < u(x,t) < 1, \quad \text{in } \Omega\times[0,T] \\ u(x,0) = u_0, \quad x\in\Omega \\ u'(x,0) = u_1, \quad x\in\Omega \\ u(x,t) = 0, \quad \Delta u(x,t) - d\dfrac{\partial u(x,t)}{\partial \nu} = 0, \quad \text{on } \partial\Omega\times[0,T] \end{cases} \tag{21}$$

where $\Omega \subset \mathbb{R}^N$, $1 \le N \le 3$, is an open bounded set with the boundary smooth enough (ν is the normal outward to the boundary $\partial\Omega$, and setting

$$G(\beta,\gamma,u) := -\left[\beta \int_\Omega |\nabla u(x,t)|^2\,dx + \gamma\right]\Delta u$$

$$H(\lambda(t),\chi,p(x),u) \quad := \quad \frac{\lambda(t)p(x)}{[1-u(x,t)]^\sigma\left[1+\chi\displaystyle\int_\Omega \frac{1}{[1-u(x,t)]^{\sigma-1}}\,dx\right]^\sigma},$$

we have shown

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^n$, $1 \le n \le 3$, be an open bounded set with diameter small enough, let $\sigma \ge 2$, β,γ,χ be nonnegative constants and $0 \le d < d_0$, where d_0 is the first boundary eigenvalue of of biharmonic operator unedr Steklov boundary conditions. Let p, c be bounded functions and $\lambda \in C^1(0,T)$ such that $\|\lambda\|_\infty < \lambda^*$, $u_0 \in H^2 \cap H_0^1(\Omega)$ (satisfying suitable compatibility asumptions) and $u_1 \in L^2(\Omega)$. Then the problem (21) has one and only one solution $u \in C^0([0,T];H^2(\Omega)) \cap C^1([0,T];L^2(\Omega))$. The same conclusion holds if $d = \infty$ e Ω is a ball.*

We assume $\sigma \geq 2$ (in the case of a Coulomb potential in the capacitor one has $\sigma = 2$, for constants $\beta, \gamma, \chi \geq 0$ which are respectively connected to self-stretching forces, tension forces and capacitance properties of the MEMS device, and for bounded real functions c, p, λ which are respectively related to anisotropic damping phenomena, permittivity profile of the constitutive material and the drop voltage applied between the ground plate at height one and the plate whose displacement is governed by the function $u(x, t)$ We assume in (21) Steklov boundary conditions, with nonnegative parameter d, accordingly to applications which demand more flexible conditions than Navier's, corresponding to $d = 0$ and Dirichlet conditions $u = u_\nu = 0$, obtained formally by setting $d = \infty$.

In the stationary case (see D. Cassani, L. Fattorusso, A. Tarsia [6]) we has shown following result

Theorem 6.2. *Let the dimension n of the space be strictly less than* 8*, then there exist λ^*, $d_0 \in (0, +\infty)$ such that for any $\lambda \in (0, \lambda^*)$ the problem has one and only one solution $u \in H^4(\Omega)$ if the diameter of Ω is small enough and besides*

(a) $0 \leq d < d_0$,

or

(b) $d = +\infty$ *and* Ω *is a ball.*

A further proof of the efficacy of our tecniques is supplyed by observing that the papers on the MEMS published before of our, though they have been written by very importan research worker studied only the stationary case in the simple form, thet is $\beta = \gamma = \chi = 0$ (see D. Cassani, J. M. d'Ó, N. Ghoussoub, [5]).

Then the complexity of problems that describe the electrostatic actuation in MEMS isn't apparent but is substantial.

7 Unexpected results

We use the adjective "unexpected" for some results that we are working in progress. These deal with problems and theorems that it seems, so far, does not be beyond the reach of the "near operators theory", that is nonuniqueness or nonexistence of solutions. For example, it seems, we can obtain results like these conteined in the famous paper of Brezis-Nirenberg (see H. Brezis, L. Nirenberg, [1]) as the following

$$\begin{cases} \Delta u(x) = -\lambda\,[1 + u(x)]^p, \text{ a. e. in } \Omega, \\ \\ u > 0, \text{ a. e. in } \Omega, \\ \\ u = 0, \text{ a. e. in } \Omega, \end{cases} \tag{22}$$

where $p = \frac{n+2}{n-2}$, $\lambda > 0$. In this case, by well known variational methods, they shown that there exists a constant $\overline{\lambda} > 0$ such that for any $\lambda \in (0, \overline{\lambda})$ the problem has at least two solutions. Instead there is one and only one solution if $\lambda = \overline{\lambda}$ or don't is any if $\lambda > \overline{\lambda}$.

We think that is a wild, or idle, idea to prove this theorem by Campanato's theory, that insted allows us to consider the following problem

$$\begin{cases} F(x, D^2u(x)) = -\mu(x)\,[1 + u(x)]^p, \text{ a. e. in } \Omega, \\ u > 0, \text{ a. e. in } \Omega, \\ u = 0, \text{ a. e. in } \Omega, \end{cases}$$

where $p = \frac{n+2}{n-2}$, $n = 3$, $\mu(x) \in L^\infty(\Omega)$ and $\mu(x) > 0$ a. e. in Ω, and we prove the theorem (see A. Iacopetti, A. Tarsia [13]):

Theorem 7.1. *Let F be measurable in x, and continuous respect to the second variable. If Condition A holds, then a positive number λ_1 exists such that if $\mu \leq \lambda_1$ then the problem* **the problem has two solutions** *in* $H^2(\Omega) \cap H_0^1(\Omega)$.

We remark that the principal part of the operator is of nonvariational type, so it is not possible to associate a functional to the equation as it happens in the case of problem (22) and to study it with variational tecniques as Mountain pass and connected arguments. Then the **"the simple math" in the Campanato meaning** becomes more and more, in these cases his utility.

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